Application of Laplace Transformation to ODE :-

* General Theory :-

let $ay'' + by' + cy = f(t)$, $y(0) = \alpha$, $y'(0) = \beta$ be a given initial value problem.

Taking Laplace transformation on both sides, we will generate an equation of the form —

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow a[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + b[s\mathcal{L}\{y\} - y(0)] + c\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

Which can be transformed to an equation of the form

$$\mathcal{L}\{y\} = g(s), \quad [\text{using } y(0) = \alpha \text{ and } y'(0) = \beta]$$

$$\Rightarrow y = \mathcal{L}^{-1}\{g(s)\}$$

Which is the required solution.

Examples:

1. Solve $y'' + y = t$, $y(0) = 1$, $y'(0) = -2$ using Laplace transformation.

Soln: Given equation is —

$$y'' + y = t, \longrightarrow (1)$$

Taking Laplace transformation on both sides of (1), we have —

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{1}{s^2} \quad \left[\text{as } \mathcal{L}\{t\} = \frac{1}{s^2}\right]$$

$$\Rightarrow (s^2+1)\mathcal{L}\{y\} = s - 2 + \frac{1}{s^2}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{s-2}{s^2+1} + \frac{1}{s^2(s^2+1)}$$

PTO.

$$\Rightarrow \mathcal{L}\{y\} = \frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

[using partial fraction]

$$\Rightarrow y = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{3}{s^2+1} \right\}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+1} \right\} + \mathcal{L}\left\{ \frac{1}{s^2} \right\} - 3\mathcal{L}\left\{ \frac{1}{s^2+1} \right\}$$

[as $\mathcal{L}^{-1}$ is linear]

$$\Rightarrow y = \cos t + t - \underline{3\sin t}$$

2. Solve $y'' - 3y' + 2y = 4e^{2t}$, $y(0) = -3$, $y'(0) = 5$ using Laplace transformation.

Sol^n: Given equation is –

$$y'' - 3y' + 2y = 4e^{2t} \longrightarrow (1)$$

Taking Laplace transformation on both sides of (1), we have –

$$\cancel{\mathcal{L}\{y''\} - 3\{}$$

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{4e^{2t}\}$$

$$\Rightarrow \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{e^{2t}\}$$ [as $\mathcal{L}$ is linear]

$$\Rightarrow s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) - 3s\mathcal{L}\{y\} + 3y(0) + 2\mathcal{L}\{y\} = \frac{4}{s-2}.$$

$$\Rightarrow (s^2 - 3s + 2)\mathcal{L}\{y\} + 3s - 14 = \frac{4}{s-2}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{4}{(s-2)(s^2-3s+2)} + \frac{14-3s}{s^2-3s+2}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}.$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left\{\frac{-7}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2}\right\}$$

$$\Rightarrow y = -7e^{t} + 4e^{2t} + 4te^{2t}.$$