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Sequence and Series of Functions.

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Introduction:

Let $n \in \mathbb{N}$ and $D \subseteq \mathbb{R}$. Let us suppose for each $n \in \mathbb{N}$ we define a function $f_n : D \to \mathbb{R}$, then the sequence—

$$\{f_1, f_2, f_3, \cdots, f_n, \cdots\} := \{f_n\}_{n=1}^{\infty}$$

is called a sequence of real valued functions defined on the set D.

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions defined on a set D, then the series—

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \cdots + f_n + \cdots$$

is called a series of real valued functions defined on the set D.

Note: If for each $n \in \mathbb{N}$ f_n is a constant function, then $\{f_n\}_{n=1}^{\infty}$ will become a sequence of real numbers. Similarly, $\sum_{n=1}^{\infty} f_n$ will become an infinite series of real numbers.

For example,

If $f_n(x) = \frac{1}{n}$, $\forall x \in D$ and $\forall n \in \mathbb{N}$.

then, $\{f_n\}_{n=1}^{\infty} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, which is a sequence of real numbers, and

$$\sum_{n=1}^{\infty} f_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots,$$ which is an infinite series of real numbers.

Example: (Sequence and Series of functions):

1. $\{f_n\}_{n=1}^{\infty}$ where

   $f_n(x) = n^2 x (1-x)^n, \; x \in \mathbb{R}, \; n = 1, 2, 3, \dots$

   is a sequence of functions defined on $\mathbb{R}$.

2. $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}, \; x \in \mathbb{R}$

   is a series of functions. [Here, $f_n(x) = \frac{x}{n(1+nx^2)}$]

3. $\{f_n\}_{n=1}^{\infty}$, where,

   $f_n(x) = nx e^{-nx}, \; x \geq 0.$

4. $\{f_n(x)\}_{n=1}^{\infty}$, where

   $f_n(x) = \frac{x}{1+nx^2}, \; x \in \mathbb{R}.$

Definition:

I. **Sequence of functions:**

Let $A \subseteq \mathbb{R}$ and let for each $n \in \mathbb{N}$ there exists a function $f_n : A \to \mathbb{R}$, then $\{f_n\}_{n=1}^{\infty}$ or $\langle f_n \rangle$ is called a sequence of functions on A.

For each $x \in A$, the sequence $\{f_n\}$ gives rise to a sequence of real numbers $\{f_n(x)\}_{n=1}^{\infty}$.

For example, If $\{f_n(x)\} = \left\{\frac{x^2}{1+n^2}\right\}_{n=1}^{\infty}$, then

$\{f_n(0)\} = \left\{\frac{0}{1+n^2}\right\}_{n=1}^{\infty} = \{0, 0, 0, \dots\}$ [The zero seq.],

$\{f_n(1)\} = \left\{\frac{1}{1+n^2}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \dots\right\}$. etc.

II. Series of functions:

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions on A, then the series $\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \cdots$ is called a series of functions on A.

For each $x \in A$, the series $\sum_{n=1}^{\infty} f_n$ gives rise to a real infinite series $\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$

For example, if $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1} \frac{n^2 x^2}{1+n^2}$, then

$$\sum_{n=1}^{\infty} f_n(0) = 0 + 0 + 0 + \cdots$$

$$\sum_{n=1}^{\infty} f_n(1) = \frac{1}{2} + \frac{4}{5} + \frac{9}{10} + \cdots \quad \text{etc.}$$

* Pointwise and Uniform Convergence:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on a set A. Let $f: A \to \mathbb{R}$ be a function. We say the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise on A to f if for each $x \in A$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ in $\mathbb{R}$.

In such case the function f is called the limit of the sequence $\{f_n\}_{n=1}^{\infty}$ and we write

$$\boxed{\lim_{n\to\infty} f_n(x) = f(x), \quad x \in A}$$

P.T.O.

Similarly, if $\sum_{n=1}^{\infty} f_n(x)$ converges for each $x \in A$ and $\sum_{n=1}^{\infty} f_n(x) = f(x)$, $\forall x \in A$, then the function $f(x)$ is called the ~~Limit~~ sum of the series $\sum_{n=1}^{\infty} f_n$.

Questions: The following questions arises:

(A) If each function f_n of a sequence $\{f_n\}$ is continuous on A, is the limit f is continuous on A.

(B) Is limit of the integral is equal to the integral of the limit.

(C) Is ~~limit~~ of the differential is equal to the differential of the limit.

The above questions can be framed as followings:

(A) Is $\lim_{x \to x_0} \left\{ \lim_{n \to \infty} f_n(x) \right\} = \lim_{n \to \infty} \left\{ \lim_{x \to x_0} f_n(x) \right\}$.

(B) Is. $\lim_{n \to \infty} \left\{ \int_a^b f_n(x)\,dx \right\} = \int_a^b \left\{ \lim_{n \to \infty} f_n(x) \right\} dx$

(C) $\lim_{n \to \infty} \left[\frac{d}{dx} \{f_n(x)\} \right] = \frac{d}{dx} \left[\lim_{n \to \infty} f_n(x) \right]$

THE ANSWER IS **NO.** (i.e. We can not interchange the limit with limit, integration and differentiation.

Counterexamples:

(A) Consider the sequence—

$$\{f_n\}_{n=1}^{\infty}, \text{ where } f_n(x) = \frac{x^{2n}}{1+x^{2n}}, \ x \in \mathbb{R}.$$

then clearly $f_n(x)$ is continuous for each $x \in \mathbb{R}$, but the limit $f(x)$, where

$$f(x) = \begin{cases} 0, & \text{if } |x| \leq 1 \\ \frac{1}{2}, & \text{if } |x| = 1 \\ 1, & \text{if } |x| \geq 1 \end{cases}$$

is not continuous on $\mathbb{R}$, as it is not continuous at $x = 1$.

(B) Consider the sequence of functions—

$$f_n(x) = n^2 x(1-x)^n, \ x \in \mathbb{R}, \ n = 1, 2, 3, \ldots$$

If $0 \leq x \leq 1$ then

$$f(x) = \lim_{n \to \infty} f_n(x) = 0 \quad \text{and so}$$

$$\int_0^1 f(x)\,dx = 0$$

But

$$\int_0^1 f_n(x)\,dx = \int_0^1 n^2 x(1-x)^n = n^2 \int_0^1 x(1-x)^n$$

$$= \cancel{n^2 \beta(2,n)} = \cancel{n^2 \frac{\Gamma(1)\Gamma(n)}{\Gamma(n+1)}}$$

$$= n^2 \beta(2, n+1)$$

$$= \frac{n^2}{(n+1)(n+2)}$$

$\therefore \lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$

Hence, $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\lim\limits_{n\to\infty} f_n(x)\right) dx$ //

(c) Consider the sequence of functions —

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}, \quad x \in \mathbb{R},\ n = 1, 2, 3, \ldots$$

then $\lim\limits_{n\to\infty} \dfrac{\sin(nx)}{\sqrt{n}} = 0 \quad \forall\, x \in \mathbb{R}$.

But $\dfrac{d}{dx}\{f_n(x)\} = \sqrt{n}\cos(nx)$.

and $\lim\limits_{n\to\infty} \left[\dfrac{d}{dx}\{f_n(x)\}\right]$ does not exist.

$$\therefore \frac{d}{dx}\left[\lim_{n\to\infty} f_n(x)\right] \neq \lim_{n\to\infty}\left[\frac{d}{dx}\{f_n(x)\}\right]$$ //.

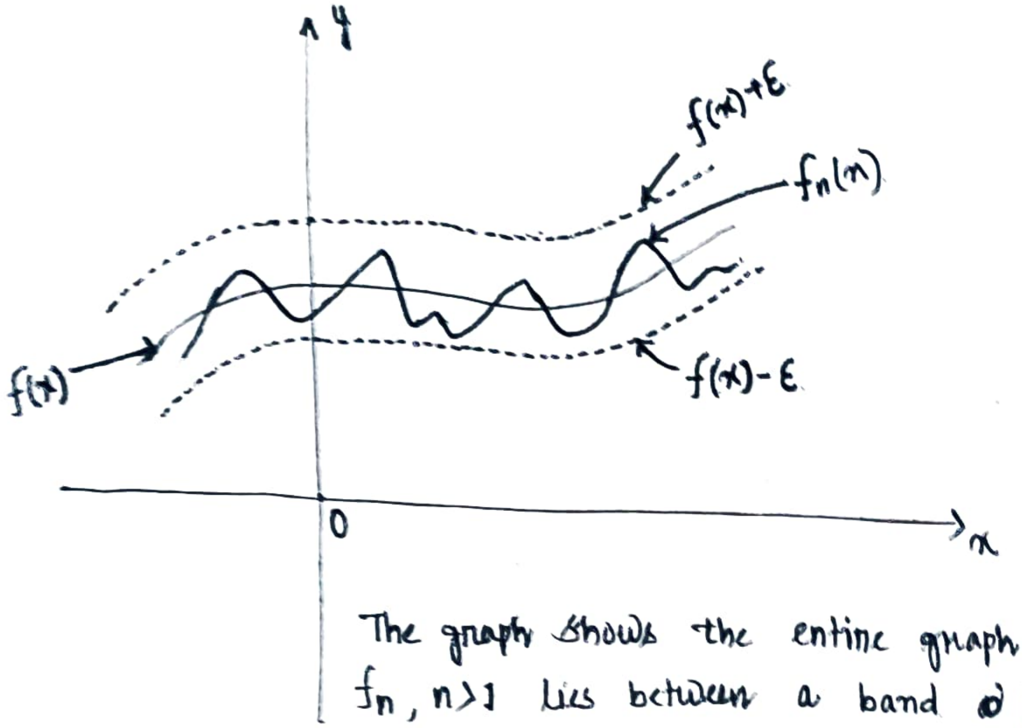
* Uniform Convergence:

* **Sequence:-** A sequence of function $\{f_n\}$ is said to converge uniformly to a function f on a set A if for each $\varepsilon > 0$ and for all x, there exists a natural number N, such that —

$$|f_n(x) - f(x)| < \varepsilon$$

i.e. $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall\, n \geq N$ and $\forall\, x \in A$.

* Geometrical Interpretation of uniform convergence:-



The graph shows the entire graph of f_n, $n \geq 1$ lies between a band (tube) of height 2ε, situated symmetrically around the graph of f.

Series:

A series $\sum_{n=1}^{\infty} f_n$ is said to converge uniformly on a set A, if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ defined as $S_n = \sum_{i=1}^{n} f_i$ converges uniformly on A.

i.e. The sequence of partial sums $\{S_n\}$ is obtained by adding the first n functions $(n \geq 1)$ of the sequence of functions.

Theorem: Every uniformly convergent sequence is pointwise convergent but the converse is not true.

Proof: let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions which converges uniformly on a set A.

$\therefore$ For every $\epsilon > 0$ ~~and for all $x \in A$~~ ~~depending on~~, $\exists$ a positive integer N ~~such that~~ – (depending on ϵ only)

$$|f_n(x) - f(x)| < \epsilon \quad \forall\ n > N \rightarrow (1)$$

$\because$ (1) is true for all $x \in A$

$\therefore$ $|f_n(x) - f(x)| < \epsilon, \ \forall\ n > N$ and $x \in A$

i.e. the sequence converges pointwise to f.

Converse is not true:

(I) Consider the sequence $\left\{\frac{1}{nx+1}\right\}_{n=1}^{\infty}$, $0 < x < 1$

Clearly $f(x) = \lim_{n \to \infty} \frac{1}{nx+1} = 0$

$\therefore$ The sequence converges pointwise to $f(x) = 0, \forall x \in (0,1)$.

let if possible the sequence be uniformly convergent on $(0,1)$, then for each $\epsilon > 0$ and $x \in (0,1)$ $\exists\ N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon.$$

$\Rightarrow \left| \frac{1}{1+nx} - 0 \right| < \varepsilon, \ \forall \ n > N$

$\Rightarrow \frac{1}{1+nx} < \varepsilon, \ \forall \ n > N$

positive.

$\Rightarrow \frac{1}{nx} < \varepsilon, \ \forall \ n > N$ [$\because$ ε is any arbitrary positive real number]

$\Rightarrow n > \frac{1}{x\varepsilon}$

$\because$ N depends on both x and ε, therefore the sequence is not uniformly convergent. //.

(II) $f_n(x) = \frac{1}{x+n}$, $x \in [0,1]$

Clearly $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x+n} = 0$

pointwise

$\therefore$ the sequence converges pointwise to $f(x) = 0$, $\forall \ x \in [0,1]$.

if the sequence is uniformly convergent then for every $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that for all $x \in [0,1]$

$|f_n(x) - f(x)| < \varepsilon$

$\Rightarrow \left| \frac{1}{n+x} - 0 \right| < \varepsilon$

$\Rightarrow \frac{1}{n+x} < \varepsilon$

$\Rightarrow n+x > \frac{1}{\varepsilon}$

$\Rightarrow n > \frac{1}{\varepsilon} - x.$

$\because$ N depends both on ε and x, therefore the sequence is not uniformly convergent. //.

Definition: (point of non-uniform convergence).

A real number c for which a sequence of real functions is not uniformly convergent ~~is called a point of non-uniform convergence~~. in an interval containing that point is called a point of non-uniform convergence.

Example: Consider the sequence—

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad 0 \le x \le a, \; a>0$$

If $x=0$, then $f_n(x)=0$, and

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0 \text{ [By L'Hopital rule]}$$

$$\therefore f(x)=0$$

If the sequence is uniformly convergent on the given interval, then for each $\varepsilon>0$ and x, there exists $N\in\mathbb{N}$ such that—

$$\left|\frac{nx}{1+n^2x^2}-0\right| < \varepsilon$$

$$\Rightarrow \frac{nx}{1+n^2x^2} < \varepsilon$$

$$\Rightarrow n^2x^2 - \frac{nx}{\varepsilon} + 1 > 0$$

$$\Rightarrow nx > \frac{1}{2\varepsilon} + \frac{1}{2}\sqrt{\frac{1}{\varepsilon^2}-4}$$

$$\Rightarrow n > \frac{1}{x}\left\{\frac{1}{2\varepsilon} + \frac{1}{2}\sqrt{\frac{1}{\varepsilon^2}-4}\right\}$$

The RHS becomes infinite if $x=0$ and hence we can not find such $N\in\mathbb{N}$, therefore $x=0$ is a point of non-uniform convergence.