Exponential Decay Model:

Let us consider a radioactive material emitting α-particles, β-particles or photons while decaying. The compartmental model for the process is —

$$\boxed{\text{Radioactive Material}} \xrightarrow{\text{Emitted particles}} \qquad \longrightarrow (1)$$

optional

We assume the following while develop the model,

1. The mass of the material is large enough so that any random fluctuation can be ignored.
2. The process is continuous.
3. The rate of decay is fixed.
4. There is no increase in mass.

By Balance Law, we have [from (1)]

$$\left\{\begin{array}{l}\text{rate of change of}\\ \text{the radioactive material}\\ \text{at any time } t\end{array}\right\} = -\left\{\begin{array}{l}\text{rate amount of}\\ \text{radioactive material}\\ \text{decayed}\end{array}\right\}$$

Let $N(t)$ be the mass of the particle at any time t and let Δt be a small change in time. Let 'K' be the rate of decay per nuclei per unit time. Therefore the number of particles (mass) at time $t+\Delta t$ is —

$$N(t+\Delta t) = N(t) - KN(t)\Delta t$$

$$\Rightarrow \frac{N(t+\Delta t) - N(t)}{\Delta t} = -KN(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} = -\lim_{\Delta t \to 0} KN(t) \quad \left[\text{taking limit on both sides}\right]$$

$$\Rightarrow \boxed{\frac{dN(t)}{dt} + KN(t) = 0}$$

Which is the required differential equation for the exponential/radioactive decay model.

Solution: The equation for the radioactive/exponential decay model is —

$$\frac{dN}{dt} + KN = 0$$

$$\therefore \quad \frac{dN}{N} = -K\,dt$$

$$\Rightarrow \int \frac{dN}{N} = -K\int dt$$

$\Rightarrow \log N = -Kt + C$, where C is an arbitrary constant.

$$\Rightarrow N(t) = e^{-Kt+C} = e^{c}e^{-Kt} = Ae^{-Kt} \quad [A = e^{c}] \rightarrow (2)$$

If 'n_0' is the initial mass, i.e. $N(0) = n_0$, then

$$N(0) = A.e^{-K.0}$$

$$\Rightarrow A = n_0$$

$$\therefore \boxed{N(t) = n_0 e^{-Kt}}$$

Note: If 'n_0' is the mass at any time t_0, i.e. if $N(t_0) = n_0$, then putting $t = t_0$ in (2) we have.

$$N(t_0) = Ae^{-Kt_0}$$

$$\Rightarrow A = n_0 e^{Kt_0}$$

$$\therefore \boxed{N(t) = n_0 e^{-K(t-t_0)}}$$

Half-~~time~~ Life of a radioactive material:

The half-life of a radioactive material is the time required to decay half of the nuclei (mass). It is denoted by τ.

Problem: If τ is the half life of a radioactive material, then find the value of K.

Solⁿ: For any radioactive material, we know that –

$$N(t) = n_0 e^{-K(t-t_0)} \rightarrow (1)$$

Where, 'n_0' is the mass at time t_0.

$\because$ τ is the half-life,

$$\therefore \frac{N(t+\tau)}{N(t)} = \frac{1}{2} \rightarrow (2)$$

Also, from (1), we have.

$$N(t+\tau) = n_0 e^{-K(t+\tau-t_0)} \rightarrow (3)$$

from ①, ② and ③, we have—

$$\frac{n_0 e^{-\kappa(t+\tau-t_0)}}{n_0 e^{-\kappa(t-t_0)}} = \frac{1}{2}$$

$$\Rightarrow e^{-\kappa(t+\tau-t_0-t+t_0)} = \frac{1}{2}$$

$$\Rightarrow e^{-\kappa\tau} = \frac{1}{2}$$

$$\Rightarrow e^{\kappa\tau} = 2$$

$$\Rightarrow \kappa\tau = \ln 2$$

$$\Rightarrow \kappa = \frac{\ln 2}{\tau}.$$