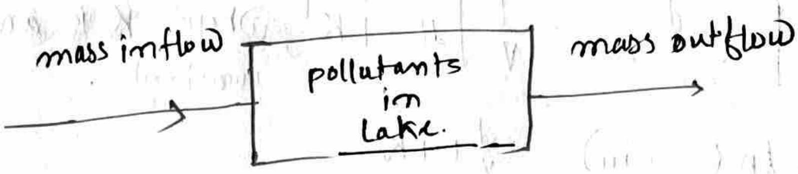
Lake pollution Model:

In lake pollution model the compartment is the lake, input is pollutants dumped into the lake, and the output is water that flows out of the lake carrying some pollutants. The compartment diagram is as follows:



By Balance Law—

$$\left\{\begin{matrix}\text{rate of change of}\\ \text{mass in lake}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate mass}\\ \text{enters lake}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate mass}\\ \text{leaves lake}\end{matrix}\right\}$$

Let 'V' be the volume and $C(t)$ be the concentration of pollutants in the lake at any time. We assume that the volume of the lake remains constant throughout the process.

$$\therefore \left\{\begin{matrix}\text{Flow of mixture}\\ \text{into lake}\end{matrix}\right\} = \left\{\begin{matrix}\text{flow of mixture}\\ \text{out of the Lake}\end{matrix}\right\} = \text{'}F\text{'} \text{ (say)}$$

Let '$M(t)$' be the mass of the pollutant at any time 't' in the lake and 'C_{in}' be the concentration of the pollutant entering the lake. Therefore by Balance law,

$$M'(t) = FC_{in} - F \cdot \frac{M(t)}{V} \rightarrow (1)$$

Also, $M(t) = C(t)V$

$$\Rightarrow M'(t) = c'(t) \cdot V \quad \text{[As volume is constant]} \rightarrow (2)$$

From (1) and (2) we have—

$$c'(t) = \frac{F}{V} C_{in} - \frac{F}{V} c(t)$$

$$\Rightarrow \frac{dC(t)}{dt} = \frac{F}{V}(C_{in} - c(t))$$

Which is the required differential equation for the lake pollution model.

Solution: The differential equation for the lake pollution model is

$$\frac{dC}{dt} = \frac{F}{V}(C_{in} - C)$$

$$\Rightarrow \frac{dC}{C_{in} - C} = \frac{F}{V} dt$$

$$\Rightarrow \int \frac{dC}{C_{in} - C} = \frac{F}{V}\int dt + K$$, where K is an arbitrary constant.

$$\Rightarrow -\ln(C - C_{in}) = \frac{F}{V} t + K.$$

$$\Rightarrow \ln(C - C_{in}) = -\frac{F}{V} t - K$$

$$\Rightarrow C = C_{in} - e^{-F/V\, t - K}.$$

$$\Rightarrow \boxed{C(t) = C_{in} - e^{-F/V\, t - K}} \longrightarrow (A)$$

Note: (IVP).

If $C(0) = C_0$ is the initial concentration, then we have

$$C(0) = C_{in} - e^{-F/V \cdot 0 - K}$$

$$\Rightarrow e^{-K} = C_0 - C_{in}$$

$$\therefore \boxed{C(t) = C_{in} - (C_0 - C_{in}) \cdot e^{-\frac{F}{V} t}}$$

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Note: (In extreme case).

taking limit in (A) as $t \to \infty$, we have.

$$\boxed{\lim_{t \to \infty} C(t) = C_{in}}$$

i.e. the concentration of the lake becomes equal to the concentration of the pollution if the process continues for a long period of time.

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Problem: How long will it take to reach the pollution level of a lake to 5 % of its initial level if only fresh water enters into the lake.

Sol^n: Given $C_{in} = 0$, as only fresh water enters lake.

$\therefore$ $C(t) = C_0 e^{-\frac{F}{V}t}$, where C_0 is the initial concentration.

$\Rightarrow t = -\frac{V}{F} \ln\left(\frac{C(t)}{C_0}\right) \longrightarrow (1)$

When the pollution level reach 5% of its initial value, we have

$C(t) = \frac{5}{100} \times C_0$

$\Rightarrow \frac{C(t)}{C_0} = 0.05 \longrightarrow (2)$

$\therefore$ from (1) and (2), we have ——

$t = -\frac{V}{F} \ln(0.05)$

$\simeq \frac{3V}{F}$.

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