Example 13.2 The utility function of a consumer is given by

$$u = 5 \log x_1 + 2 \log x_2.$$

Find out the combination of x_1 and x_2 which will maximize the utility function subject to the satisfaction of the budget constraint

$$4x_1 + 2x_2 = 28.$$

Solution: With the objective function $u = 5 \log x_1 + 2 \log x_2$ to be maximized subject to the budget constraint

$$4x_1 + 2x_2 = 28,$$

the Lagrangean function is defined as

$$L = 5 \log x_1 + 2 \log x_2 + \lambda(28 - 4x_1 - 2x_2).$$

The first order condition of maximization requires

$$L_{x_1} = \frac{\partial L}{\partial x_1} = 0; \; L_{x_2} = \frac{\partial L}{\partial x_2} = 0 \text{ and } L_\lambda = \frac{\partial L}{\partial \lambda} = 0.$$

Now

$$\left.\begin{aligned} L_{x_1} &= \frac{5}{x_1} - 4\lambda = 0 \\ L_{x_2} &= \frac{2}{x_2} - 2\lambda = 0 \\ L_\lambda &= 28 - 4x_1 - 2x_2 = 0 \end{aligned}\right\} \quad (13.19)$$

From first two equations of equation (13.19), we have

$$\lambda = \frac{5}{4x_1} = \frac{2}{2x_2}$$

or

$$4x_1 - 5x_2 = 0 \quad (13.20)$$

From the third equation of equation (13.19)

$$4x_1 + 2x_2 = 28. \quad (13.21)$$

Subtracting equation (13.20) from equation (13.21), we get

$$7x_2 = 28.$$

$$\therefore \quad \bar{x}_2 = 4.$$

Now substituting $\bar{x}_2 = 4$ in equation (13.21), we get

$$4x_1 = 20. \quad \therefore \quad \bar{x}_1 = 5.$$

The second order condition of maximization requires that $|\bar{H}_2| > 0$.

Now from equation (13.19) and the budget constraint

$$L_{xx} = -\frac{5}{x_1^2}; \; L_{xy} = 0 = L_{yx}; \; L_{yy} = \frac{2}{x_2^2}$$

and

$$g_1 = 4; g_2 = 2$$

$$\therefore \quad |\bar{H}_z| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{xx} & L_{xy} \\ g_2 & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 4 & 2 \\ 4 & -\frac{5}{x_1^2} & 0 \\ 2 & 0 & -\frac{2}{x_2^2} \end{vmatrix}$$

For $x_1 = 5$ and $x_2 = 4$, we get

$$|\bar{H}_z| = \begin{vmatrix} 0 & 4 & 2 \\ 4 & -\frac{1}{5} & 0 \\ 2 & 0 & -\frac{1}{8} \end{vmatrix} = 2 + \frac{4}{5} = \frac{14}{5} > 0.$$

Since $|\bar{H}_2| > 0$, $\bar{x}_1 = 5$ and $\bar{x}_2 = 4$ will maximize the utility of the consumer. The maximum utility will be

$$\bar{u} = 5 \log (5) + 2 \log (4)$$
$$= 4.7$$

Example 13.7 A producer desires to minimize the cost of production

$$C = 16k + 4L$$

where k and L are capital and labour respectively subject to the given production function

$$Q = 5k^{\frac{1}{2}}L^{\frac{1}{2}}.$$

Find out the equilibrium combination of inputs (k and L) in order to minimize the cost of production when output $Q = 40$.

Solution: In order to minimize the cost function

$$C = 16k + 4L$$

subject to the output constraint $Q = 5k^{\frac{1}{2}}L^{\frac{1}{2}} = 40$, we first construct the Lagrange function Z such that

$$Z = 16k + 4L + \lambda\left[40 - 5k^{\frac{1}{2}}L^{\frac{1}{2}}\right]. \tag{13.57}$$

First order condition of minimization requires that

$$Z_L = \frac{\partial Z}{\partial L} = 0; \; Z_k = \frac{\partial Z}{\partial k} = 0 \text{ and } Z_\lambda = \frac{\partial Z}{\partial \lambda} = 0.$$

Now

$$Z_L = 4 - \lambda\frac{1}{2}5k^{\frac{1}{2}}L^{\frac{1}{2}-1} = 4 - \frac{1}{2}\lambda\frac{Q}{L} = 0 \tag{13.58}$$

$$Z_k = 16 - \lambda\frac{1}{2}5k^{\frac{1}{2}-1}L^{\frac{1}{2}} = 16 - \frac{1}{2}\lambda\frac{Q}{k} = 0 \tag{13.59}$$

$$Z_\lambda = 40 - 5k^{\frac{1}{2}}L^{\frac{1}{2}} = 0 \tag{13.60}$$

From equations (13.58) and (13.59)

$$\lambda = \frac{8L}{Q} = \frac{32k}{Q} \qquad (13.61)$$

or

$$8L = 32k. \quad \therefore \quad L = 4k.$$

Now substituting $L = 4k$ in equation (13.60), we get

$$40 = 5k^{\frac{1}{2}}(4k)^{\frac{1}{2}} = 5\sqrt{4}k^{\frac{1}{2}+\frac{1}{2}}$$

or

$$40 = 10k.$$

$\therefore$

$$\bar{k} = 4.$$

$\therefore$

$$\bar{L} = 4k = 16.$$

Now to get the value of λ, we substitute $L = 16$ in equation (13.61).

$$\lambda = \frac{8 \times 16}{40} = \frac{16}{5}. \qquad (13.62)$$

To test whether $\bar{L} = 16$ and $\bar{k} = 4$ will minimize the cost function or not, we go for the second order condition which requires that $|\bar{H}_2| < 0$ where

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_L & g_k \\ g_L & Z_{LL} & Z_{Lk} \\ g_k & Z_{kL} & Z_{kk} \end{vmatrix}$$

Now from equation (13.60),

$$g_L = -\frac{1}{2}5k^{\frac{1}{2}}L^{\frac{1}{2}-1} = -\frac{1}{2}\frac{Q}{L} = -\frac{1}{2}\frac{40}{16} = -\frac{5}{4}$$

$$g_k = -\frac{1}{2}5k^{\frac{1}{2}-1}L^{\frac{1}{2}} = -\frac{1}{2}\frac{Q}{k} = -\frac{1}{2}\frac{40}{4} = -5$$

From equations (13.58) and (13.59)

$$Z_{LL} = -\left(\frac{1}{2}-1\right)\lambda\frac{1}{2}5k^{\frac{1}{2}}L^{\frac{1}{2}-1-1} = \frac{1}{4}\lambda\frac{Q}{L^2} = \frac{1}{4} \times \frac{16}{5} \times \frac{40}{16^2} = \frac{1}{8}$$

$$Z_{Lk} = -\frac{1}{2}\lambda\frac{1}{2}5k^{\frac{1}{2}-1}L^{\frac{1}{2}-1} = -\frac{1}{4}\lambda\frac{Q}{kL}$$

$$= -\frac{1}{4} \times \frac{16}{5} \times \frac{40}{4 \times 16} = -\frac{1}{2}$$

$$Z_{kL} = -\frac{1}{2}\lambda\frac{1}{2}5k^{\frac{1}{2}-1}L^{\frac{1}{2}-1} = -\frac{1}{4}\lambda\frac{Q}{kL}$$

$$= -\frac{1}{4} \times \frac{16}{5} \times \frac{40}{4 \times 16} = -\frac{1}{2}$$

$$Z_{kk} = -\left(\frac{1}{2}-1\right)\lambda\frac{1}{2}5k^{\frac{1}{2}-1-1}L^{\frac{1}{2}} = \frac{1}{4}\lambda\frac{Q}{k^2}$$

$$= \frac{1}{4}\times\frac{16}{5}\times\frac{40}{4^2} = 2$$

$$\therefore \qquad |\bar{H}_2| = \begin{vmatrix} 0 & -\frac{5}{4} & -5 \\ -\frac{5}{4} & \frac{1}{8} & -\frac{1}{2} \\ -5 & -\frac{1}{2} & 2 \end{vmatrix}$$

$$= \frac{5}{4}\begin{vmatrix} -\frac{5}{4} & -\frac{1}{2} \\ -5 & 2 \end{vmatrix} - 5\begin{vmatrix} -\frac{5}{4} & \frac{1}{8} \\ -5 & -\frac{1}{2} \end{vmatrix}$$

$$= \frac{5}{4}\left(-\frac{10}{4}-\frac{5}{2}\right) - 5\left(\frac{5}{8}+\frac{5}{8}\right)$$

$$= -\frac{25}{4}-\frac{25}{4} = -\frac{25}{2} < 0.$$

Since $|\bar{H}_2| < 0$, $L = 16$ and $k = 4$ will minimize the cost of production subject to the output constraint. So the minimum cost will be

$$C = 16(4) + 4(16) = 128.$$

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