## Monotonic Sequence

+ **Monotonic increasing and decreasing sequences:**

A sequence $\langle u_n \rangle$ is said to be a monotonically increasing sequence if

$$u_{n+1} \geq u_n, \forall n \in N$$

Similarly $\langle u_n \rangle$ is said to be a monotonically decreasing sequence if

$$u_{n+1} \leq u_n, \forall n \in N$$

**For example:-**

the sequence $\langle n^2 \rangle$. i.e $1, 4, 9, 16, \ldots$ is an increasing sequence.

And

the sequence $\langle \frac{1}{n} \rangle$ i.e, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a decreasing sequence.

**Example-(1)** Show that the sequence $\langle u_n \rangle$, where $u_n = \frac{1}{n}$ is monotonically decreasing.

**Soln:** $u_n = \frac{1}{n}$

$$\therefore u_{n+1} = \frac{1}{n+1}$$

$$\therefore u_{n+1} - u_n = \frac{1}{n+1} - \frac{1}{n}$$

$$= \frac{n-(n+1)}{n(n+1)}$$

$$= \frac{-1}{n(n+1)} < 0 \ \forall n \in N$$

$\therefore U_{n+1} - U_n < 0$

$\Rightarrow U_{n+1} < U_n \ \forall n \in N$

Hence $\langle U_n \rangle$ is monotonically decreasing.

<u>Example</u> - (3) Show that the sequence $\langle U_n \rangle$, where $U_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ is monotonically increasing.

<u>Soln</u> Here

$$U_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$\therefore U_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{\cancel{2n+2}} \frac{1}{2(n+1)}$$

$$\Rightarrow U_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2}$$

$$= \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

Now

$$U_{n+1} - U_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{(2n+2)\cancel{(n+1)} + }{(2n+1)\cancel{(2n+2)}(n+1)}$$

$$= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{(n+1)}$$

$$= \frac{1}{2n+1} + \frac{2-1}{2(n+1)}$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$= \frac{2n+2+2n+1}{(2n+1)(2n+2)}$$

$$= \frac{4n+3}{(2n+1)(2n+2)} > 0 \quad \forall n \in N.$$

$\therefore U_{n+1} - U_n > 0$

$\Rightarrow U_{n+1} > U_n \ \forall n \in N$

Hence $\langle U_n \rangle$ is monotonically increasing.

<u>Example 4.</u> Examine if the sequence $\left\langle U_n \,\middle|\, U_n = \frac{2n-7}{3n+2} \right\rangle$ is monotonic increasing and bounded above.

<u>Soln</u> Here

$$U_n = \frac{2n-7}{3n+2}$$

$$\therefore U_{n+1} = \frac{2(n+1)-7}{3(n+1)+2}$$

$$= \frac{2n+2-7}{3n+3+2}$$

$$= \frac{2n-5}{3n+5}$$

$\therefore\ U_{n+1} - U_n = \dfrac{2n-5}{2n+5} - \dfrac{2n-7}{3n+2}$

$= \dfrac{(2n-5)(3n+2) - (2n-7)(2n+5)}{(2n+5)(3n+2)}$

$= \dfrac{6n^2+4n-15n-10-6n^2-10n+21n+35}{(2n+5)(3n+2)}$

$= \dfrac{25}{(2n+5)(3n+2)} > 0 \ \forall n \in N$

$\therefore\ U_{n+1} - U_n > 0$

$\Rightarrow U_{n+1} > U_n \ \forall n \in N$

Hence $\langle U_n \rangle$ is monotonically increasing.

Again,

$1 - U_n = 1 - \dfrac{2n-7}{3n+2}$

$= \dfrac{3n+2-2n-7}{3n+2}$

$= \dfrac{n+9}{3n+2} > 0$

$\therefore\ 1 - U_n > 0$

$\Rightarrow 1 > U_n \ \forall n \in N$

$\Rightarrow U_n < 1 \ \forall n \in N$

$\therefore \langle U_n \rangle$ is bounded above

Hence $\langle U_n \rangle$ is monotonically increasing and bounded above.

Theorem : 1

Prove that a monotonic increasing bounded above sequence is convergent and converges to the upper bound (Supremum).

Proof: let $\langle U_n \rangle$ be a monotonic increasing and bounded above sequence.

let $\beta$ be the supremum of $\langle U_n \rangle$

$\therefore U_n \leq \beta, \forall n \in N$

Then for any given small $\epsilon > 0$, there exists atleast one value of n, say m such that

$U_m > \beta - \epsilon$

$\Rightarrow \beta - \epsilon < U_m$ —(i)

Since $\langle U_n \rangle$ is monotonic increasing, therefore

$U_m \leq U_n, \forall n \geq m$

$\therefore$ (i) $\Rightarrow \beta - \epsilon < U_n, \forall n \geq m$ —(ii)

Again, $U_n \leq \beta, \forall n \in N$

$\Rightarrow U_n < \beta + \epsilon, \forall n \in N$ —(iii)

$\therefore$ eqn —(ii) and (iii)

$\Rightarrow \beta - \epsilon < U_n < \beta + \epsilon, \forall n \geq m$

$\Rightarrow -\epsilon < U_n - \beta < \epsilon, \forall n \geq m$

$\Rightarrow |U_n - \beta| < \epsilon, \forall n \geq m$

Hence $\langle U_n \rangle$ is convergent and converges to the upper bound $\beta$.

Theorem – 2

Prove that a monotonic decreasing bounded below sequence is convergent and converges to the lower bound (infimum).

**Proof:** let $\langle u_n \rangle$ be a monotonic decreasing and bounded below sequence.

let $\beta$ be the infimum of $\langle u_n \rangle$

$\therefore u_n \geq \beta ; \forall n \in N$

Then for any given small $\epsilon > 0$, there exists atleast one value of $n$, say $m$ such that

$u_m < \beta + \epsilon$ — (i)

Since $\langle u_n \rangle$ is monotonic decreasing, therefore $u_n \leq u_m , \forall n \geq m$

$\therefore (i) \Rightarrow u_n < \beta + \epsilon , \forall n \geq m$ — (ii)

Again, $u_n \geq \beta , \forall n \in N$

$\Rightarrow u_n > \beta - \epsilon , \forall n \in N$

$\Rightarrow \beta - \epsilon < u_n , \forall n \in N$ — (iii)

eqn –(ii) and (iii)

$\Rightarrow \beta - \epsilon < u_n < \beta + \epsilon , \forall n \geq m$

$\Rightarrow -\epsilon < u_n - \beta < \epsilon , \forall n \geq m$

$\Rightarrow |u_n - \beta| < \epsilon , \forall n \geq m$

Hence the sequence $\langle u_n \rangle$ is convergent and converges to the lower bound $\beta$.

Example - 1

Show that the sequence $\langle u_n \rangle$, where $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$ is convergent.

Soln: Here,

$$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$$

$$\therefore u_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n+2}$$

$$= \frac{1}{n+1} + \frac{1}{n+3} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$\therefore u_{n+1} - u_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1}$$

$$= \frac{2(n+1) + 2n+1 - 2(2n+1)}{2(n+1)(2n+1)}$$

$$= \frac{4n+3-4n-2}{2(n+1)(2n+1)}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0, \ \forall n \in N$$

$\therefore u_{n+1} - u_n > 0$ $\cancel{\forall n \in N}$

$\Rightarrow u_{n+1} > u_n \ \forall n \in N$

$\therefore \langle u_n \rangle$ is monotonically increasing.

Again, $U_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}$

$$= \frac{n}{n+1} < 1, \forall n \in N$$

$\therefore U_n < 1, \forall n \in N$

$\Rightarrow \langle U_n \rangle$ is bounded above.

Thus $\langle U_n \rangle$ is monotonically increasing and bounded above and Hence is Convergent.

Example-(2)

Examine for convergence the following sequences; and find it's limit.

(i) $\left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \cdots\right\}$

(i) Soln Here, $U_n = \frac{2n-1}{2n}$

$$\therefore U_{n+1} = \frac{2(n+1)-1}{2(n+1)}$$

$$= \frac{2n+2-1}{2n+2}$$

$$= \frac{2n+1}{2n+2}$$

$$\therefore U_{n+1} - U_n = \frac{2n+1}{2n+2} - \frac{2n-1}{2n}$$

$$= \frac{(2n+1)(2n) - (2n-1)(2n+2)}{2n(2n+2)}$$

$$= \frac{4n^2+2n-(4n^2+4n-2n-2)}{2n(2n+2)}$$

$$= \frac{\cancel{4n^2}+\cancel{2n}-\cancel{4n^2}-\cancel{2n}+2}{2n(2n+2)}$$

$$= \frac{2}{2n(2n+2)} > 0 \quad \forall\, n \in N$$

$\therefore\ u_{n+1} - u_n > 0$

$\Rightarrow u_{n+1} > u_n,\ \forall\, n \in N$

Thus $\langle u_n \rangle$ is monotonically increasing.

Again,

$$1 - u_n = 1 - \frac{2n-1}{2n}$$

$$= \frac{2n-2n+1}{2n}$$

$$= \frac{1}{2n} > 0$$

$\Rightarrow 1 - u_n > 0$

$\Rightarrow u_n < 1,\ \forall\, n \in N$

$\therefore \langle u_n \rangle$ is bounded above

Thus $\langle u_n \rangle$ is monotonically increasing and bounded above and Hence it is Convergent.

Now

To find limit

we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2n-1}{2n} = \lim_{n\to\infty} \frac{\cancel{n}(2-\frac{1}{n})}{\cancel{n}(2)}$$

$$= \frac{2}{2} = 1$$

So, $\lim_{n\to\infty} u_n = 1$ .//-