Improper Integral

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Improper Integral.

An Integral $\int_a^b f(x)dx$ is called Proper Integral if limits are finite and $f(x)$ is bounded at a or b or both. If $\int_a^b f(x)dx$ is not proper, it is called improper integral.

Ist Kind

$\int_a^b f(x)dx$, if limits are finite but $f(x)$ is Unbounded, it is called improper integral of 1st kind.

$$\int_0^2 \frac{dx}{x(x-1)}, \quad \int_1^2 \frac{dx}{(x-1)(x-2)}, \quad \int_0^4 \frac{dx}{(x-4)^2}, \quad \int_0^5 \frac{dx}{(x-2)(x+3)}$$

IInd kind

$\int_a^\infty f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$ are improper integral of IInd kind. i.e. if limits are infinite

Convergence.

$\int_a^b f(x)dx$ be a Improper integral of Ist kind and $f(x)$ unbounded at $x = a$.

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)dx.$$

= a finite no, given integral Convergent and vice versa.

If $\int_a^b f(x)dx$ unbounded at $x = b$.

$\lim_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)dx$ = a finite no, then given integral convergent.

Ex-1 $\int_0^1 \frac{dx}{x^2}$, Test the Convergence.

Sol^n:- $\int_0^1 \frac{dx}{x^2} = \lim_{\epsilon \to 0} \int_{0+\epsilon}^1 \frac{dx}{x^2}$

$= \lim_{\epsilon \to 0} \left[\frac{x^{-2+1}}{-2+1} \right]_{0+\epsilon}^1$

$= \lim_{\epsilon \to 0} \left[-\frac{1}{x} \right]_\epsilon^1$

$= -\lim_{\epsilon \to 0} \left[1 - \frac{1}{\epsilon} \right]$

$= \infty$.

Given, integral is divergent.

Ex-2 $\int_0^1 \frac{dx}{\sqrt{1-x}}$, Test the Convergence.

Sol^n:- $f(x) = \frac{1}{\sqrt{1-x}}$ is unbounded at $x = 1$.

$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x}}$

$= \lim_{\epsilon \to 0} \left[\frac{(1-x)^{-1/2+1}}{-1/2+1} (-1) \right]_0^{1-\epsilon}$

$= -\lim_{\epsilon \to 0} \left[2(1-x)^{1/2} \right]_0^{1-\epsilon}$

$= \cancel{-\frac{1}{2} \lim} = -2 \lim_{\epsilon \to 0} \left[(1-1+\epsilon)^{1/2} - 1^{1/2} \right]$

$= -2 \lim_{\epsilon \to 0} \left[\epsilon^{1/2} - 1 \right]$

$= -2 [0 - 1]$

$= 2$.

Given integral is convergent.

Ex-3 $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol^n:- $f(x) = \frac{1}{\sqrt{1-x^2}}$ is unbounded at $x=1$.

$\therefore \lim_{\epsilon\to 0} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}}$

$= \lim_{\epsilon\to 0} \left[\sin^{-1}x\right]_0^{1-\epsilon}$

$= \lim_{\epsilon\to 0} \left[\sin^{-1}(1-\epsilon) - \sin^{-1}0\right]$

$= \sin^{-1}1 - \sin^{-1}0$

$= \pi/2 - 0$

$= \pi/2$

So, given integral is Convergent.

Compa

$\int_a^b f(x)dx$, where $f(x)$ unbounded at $x = a$

If f and g are two positive functions on $[a,b]$ and $f(x) \le g(x)$.

Then, $\int_a^b f(x)dx$ convergent if $\int_a^b g(x)\,dx$ convergent.

$\int_a^b g(x)\,dx$ divergent if $\int_a^b g(x)\,dx$ divergent.

Alternative form.

If $f(x)$ and $g(x)$ are two positive functions and $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$, a finite number.

Then, $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ either convergent or divergent.

Comparison Integral.

$\int_a^b \frac{dx}{(x-a)^n}$, convergent if $n<1$.
divergent if $n \geqslant 1$.

Soln

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0} \int_{a+\epsilon}^{b} \frac{dx}{(x-a)^n}.$$

$$= \lim_{\epsilon\to 0} \left[\frac{(x-a)^{1-n}}{1-n} \right]_{a+\epsilon}^{b}.$$

$$= \lim_{\epsilon\to 0} \left[\frac{(b-a)^{1-n}}{1-n} - \frac{(\epsilon)^{1-n}}{1-n} \right].$$

Case I for $n<1$.

$n<1$.

$\Rightarrow 0<1-n$.

$\therefore \lim_{\epsilon\to 0} \epsilon^{1-n} = 0$.

$$\int_a^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n} - 0$$

$$= \frac{(b-a)^{1-n}}{1-n}$$

So, it is convergent.

Case II

$n>1$.

$\Rightarrow 1-n<0$.

$\therefore \lim_{\epsilon\to 0} \epsilon^{1-n} = \infty$.

$\therefore \int_a^b \frac{dx}{(x-a)^n} = \infty$.

So, it is divergent.

Case III if $n=1$.

$$\int_a^b f(x)dx = \int_a^b \frac{dx}{(x-a)^n}$$

$$= \int_a^b \frac{dx}{x-a}$$

$$= \lim_{\epsilon\to 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)}$$

$$= \lim_{\epsilon\to 0} \Big[\log (x-a)\Big]_{a+\epsilon}^b$$

$$= \lim_{\epsilon\to 0} \Big[\log (b-a) - \log \epsilon\Big]$$

$$= \infty.$$

So it is divergent.

Hence, $\int_a^b \frac{dx}{(x-a)^n}$ convergent if $n<1$.
divergent if $n\geq 1$.

Ex-4 Test the Convergence of the integral.

$$\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$$

Solⁿ:- $f(x) = \dfrac{1}{x^{1/3}(1+x^2)}$

Here, $f(x)$ is unbounded at $x=0$.

Consider a function $g(x) = \dfrac{1}{x^{1/3}}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{x^{1/3}(1+x^2)} \cdot x^{1/3}$$

$$= \lim_{x\to 0} \frac{1}{1+x^2}$$

$$= \frac{1}{1+0} = 1$$, a finite number.

$\therefore \int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1/3}}\,dx$. is convergent because $n = \frac{1}{3}$ which is less than 1.

So, By Comparison test the given integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.

Ex-5 Test the Convergence $\int_0^1 \frac{dx}{x^3(1+x^2)}$.

Solⁿ:- $f(x) = \dfrac{1}{x^3(1+x^2)}$ is unbounded at $x=0$.

Let us consider a function $g(x) = \frac{1}{x^3}$.

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{x^3(1+x^2)} \cdot x^3.$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{1+x^2}$$

$$= \frac{1}{1+0} = 1$$, a finite no.

$\therefore \int_0 g(x)\,dx = \int_0 \frac{1}{x^3}$ is divergent. since $n=3$ greater th...

So, by comparison test, the given integral

$$\int_0^1 \frac{dx}{x^3(1+x^2)} \text{ is divergent.}$$

(4)

Ex-6 Test the convergency $\int_1^2 \frac{dx}{\sqrt{x^4-1}}$

Sol^n:-

$$\int_1^2 \frac{dx}{\sqrt{x^4-1}} = \int_1^2 \frac{dx}{\sqrt{x-1}\,\sqrt{x+1}\,\sqrt{x^2+1}}$$

$$f(x) = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}\,\sqrt{x^2+1}} \text{ is unbounded at } x=1.$$

Consider a function $g(x) = \frac{1}{\sqrt{x-1}}$

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{1}{\sqrt{x-1}\,\sqrt{x+1}\,\sqrt{x^2+1}} \cdot \sqrt{x-1}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x+1}\,\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2}, \text{ which is a finite no.}$$

$\therefore \int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{(x-1)^{1/2}}$ is convergent because $n = \frac{1}{2}$ less than 1.

So, By comparison test, the given integral

$$\int_1^2 \frac{dx}{\sqrt{x^4-1}} \text{ is convergent.}$$

$\int_0 \sqrt{x(1-x)}$

Solⁿ:- $I = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

$= \int_0^a \frac{dx}{\sqrt{x(1-x)}} + \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$

$= I_1 + I_2$.

Test for I_1 $\quad I_1 = \int_0^a dx/\sqrt{x(1-x)}$

$f(x) = \frac{1}{\sqrt{x(1-x)}}$ is unbounded at $x=0$.

Consider a function $g(x) = \frac{1}{\sqrt{x}}$.

Now, $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{\sqrt{x}\sqrt{1-x}} \cdot \sqrt{x}$.

$= \lim_{x\to 0} \frac{1}{\sqrt{1-x}}$

$= 1$. , a finite no.

$\therefore \int_0^a g(x)\,dx = \int_0^a \frac{1}{\sqrt{x}}$ is convergent because $n = \frac{1}{2}$ less than

So, By Comparison test I_1 is convergent.

Test for I_2 $\quad I_2 = \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$

$f(x) = \frac{1}{\sqrt{x(1-x)}}$ is unbounded at $x=1$.

Consider a function $g(x) = \frac{1}{\sqrt{1-x}}$.

$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{1}{\sqrt{x}\cdot\sqrt{1-x}} \cdot \sqrt{1-x}$

$= \lim_{x\to 1} \frac{1}{\sqrt{x}}$

$= 1$. , a finite no.

$\therefore \int_a g(x)\,dx = \int_a \frac{dx}{\sqrt{\pi - x}}$ is Convergent because $n = \frac{1}{2} < 1$.

So, By Comparison test, I_2 is convergent. (5)

Since, I_1 and I_2 are convergent.

So, the given integral I is also convergent.

Ex-8 $\int_0^{\pi/2} \frac{\cos x}{x^2}\,dx$.

Sol^n

Here, $f(x) = \frac{\cos x}{x^2}$

$f(x)$ is unbounded at $x = 0$.

Consider a function $g(x) = \frac{1}{x^2}$.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\cos x}{x^2} \cdot x^2$$

$$= \lim_{x \to 0} \cos x.$$

$= 1$, a finite no.

$\therefore \int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{1}{x^2}$ is divergent integral because $n = 2$ greater than 1.

So, By comparison test ~~f(x) =~~ the given integral

$\int_0^{\pi/2} \frac{\cos x}{x^2}$ is also divergent.

Ex-9 Show that $\int_0 x^{n-1} e^{-x}\, dx$, Convergent if $n > 0$.

Here;

$$f(x) = x^{n-1} e^{-x}.$$

If $n-1 \geq 0$, Then given integral is proper integral. Then it is obviously convergent.

let,

$$0 < n < 1.$$

Consider a f^n $g(x) = x^{n-1}$.

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^{n-1} e^{-x}}{x^{n-1}}$$

$$= \lim_{x\to 0} e^{0}$$

$= 1$, a finite no.

$$\therefore \int_0^1 g(x)\, dx = \int_0^1 x^{n-1}\, dx.$$

$= \int_0^1 \frac{dx}{x^{1-n}}$ is convergent because power is less than 1 ($\because 0 < n < 1$)

So, By comparison test

$\int_0^1 f(x)\, dx$ is convergent when $0 < n < 1$.

Hence, The given integral is convergent for $n > 0$.

$\int_a^b f(x)dx$ be a improper integral of I^{st} kind where $f(x)$ unbounded at $x=a$. If there exist a number μ between 0 and 1. $(0<\mu<1)$ such that $\lim_{x\to a}(x-a)^{\mu}f(x)$ exists, then $\int_a^b f(x)dx$ convergent.

If $\mu \geq 1$ such that $\lim_{x\to a}(x-a)^{\mu}f(x)$ exists then $\int_a^b f(x)dx$ divergent. Also integration is divergent if $\lim_{x\to a}(x-a)^{\mu}f(x) = \infty$ or $-\infty$.

Ex-10 Test the convergency of the integral.

Solⁿ:-
$$I = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

Solⁿ:-
$$I = \int_0^a \frac{dx}{\sqrt{x(1-x)}} + \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$$

$$= I_1 + I_2 \quad \text{(let)}.$$

Test of I_1

$$I_1 = \int_0^a \frac{dx}{\sqrt{x(1-x)}}$$

Take $\mu = \frac{1}{2}$.

$$\lim_{x\to 0} f(x)(x-0)^{\mu} = \lim_{x\to 0} \frac{1}{\sqrt{x}\sqrt{1-x}} \cdot \sqrt{x}$$

$$= 1, \text{ limit exist.}$$

and $\mu = \frac{1}{2}$. So, by μ test.

$\therefore$ I_1 is convergent.

$$I_2 = \int_a \frac{dx}{\sqrt{x}\,\sqrt{1-x}}.$$

take, $\mu = \frac{1}{2}$.

$$\lim_{x\to 1} f(x)\,(1-x)^{\mu} = \lim_{x\to 1} \frac{1}{\sqrt{x}\sqrt{1-x}} \cdot (1-x)^{1/2}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}} = 1, \text{ limit exist}$$

and $\mu = \frac{1}{2}$.

So, by μ test I_2 is also Convergent.

Since, I_1 and I_2 are both convergent.

$\therefore$ I is convergent.

Ex-11. Test the convergency of the integral.

$$\int_0^1 x^{p-1} e^{-x}\, dx.$$

Soln :- $f(x) = x^{p-1} e^{-x}$.

If $p-1 \geq 0$ i.e. $p \geq 1$, then function is always bounde at 0 and 1. So, the integral is proper and so, it is obviously convergent.

let, $0 < p < 1$. then,

$$\lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{p-1} e^{-x} x^{\mu}$$

$$= \lim_{x\to 0} x^{p-1+\mu} e^{-x}.$$

$$= 1, \text{ if } \mu + p - 1 = 0.$$

$$\Rightarrow \mu = 1 - p.$$

For limit exist

$\mu = 1-p$.

So, for Convergency by μ test

$0 < 1-p < 1.$

i.e. $0 < p < 1.$

Hence, given integral is convergent for $p > 0$.

By μ test integration is divergent if $\mu \geq 1$.

$\Rightarrow 1-p \geq 1.$

$\Rightarrow 0 \geq p.$

Ex-12 Test the Convergency of the integral.

$$\int_0^{\pi/2} \frac{\cos x}{x^n}\, dx.$$

Soln $f(x) = \dfrac{\cos x}{x^n}$

If $n \leq 0$, given integral is Proper integral.

So, it is convergent.

d Let, $n > 0$.

$$\lim_{x\to 0} x^{\mu} \frac{\cos x}{x^n} = \lim_{x\to 0} x^{\mu-n} \cos x$$

$= 1$, if $\mu - n = 0$

$\Rightarrow \mu = n.$

$\therefore$ By μ test, given integral is convergent

$0 < \mu < 1.$

$0 < n < 1$

Hence, given integral is convergent when $n<1$.

Also, By μ test integral is divergent

if $\mu \geq 1$.

i.e. $n \geq 1$. #

Improper Integral of

Ex-13 Test the Convergency of $\int_1^{\infty} \frac{dx}{\sqrt{x}}$.

Solⁿ :-

$$\lim_{X \to \infty} \int_1^{X} \frac{dx}{\sqrt{x}}$$

$$= \lim_{X \to \infty} \left[\frac{x^{1/2}}{1/2} \right]_1^{X}$$

$$= \lim_{X \to \infty} \left[2 \left(X^{1/2} - 1 \right) \right]$$

$$= \infty.$$

∴ Given integral is divergent.

Ex-14 Test the Convergency of $\int_1^{\infty} \frac{dx}{x^{3/2}}$.

Solⁿ :-

$$\lim_{X \to \infty} \int_1^{X} \frac{dx}{x^{3/2}}$$

$$= \lim_{X \to \infty} \left[\frac{x^{-1/2}}{-1/2} \right]_1^{X}$$

$$= \lim_{X \to \infty} \left[-2 \left(X^{-1/2} - 1 \right) \right]$$

$$= -2 \lim_{X \to \infty} \left(\frac{1}{\sqrt{X}} - 1 \right)$$

$$= 2.$$

Given integral is convergent.

Test the

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$ 8

Sol^n

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$= \int_{-\infty}^{a} \frac{dx}{1+x^2} + \int_{a}^{\infty} \frac{dx}{1+x^2}$$

$$= \cancel{I_1 + I_2}$$

$$= \lim_{X\to\infty} \int_{-X}^{a} \frac{dx}{1+x^2} + \lim_{X\to\infty} \int_{a}^{X} \frac{dx}{1+x^2}$$

$$= \lim_{X\to\infty} \left[\tan^{-1} x\right]_{-X}^{a} + \lim_{X\to\infty} \left[\tan^{-1} x\right]_{a}^{X}$$

$$= \lim_{X\to\infty} \left[\tan^{-1} a - \tan^{-1}(-X)\right] + \lim_{X\to\infty} \left[\tan^{-1} X - \tan^{-1} a\right]$$

$$= \cancel{\tan^{-1} a + \pi/2 + \pi/2 - \tan^{-1} a}$$

$$= \tan^{-1} a - \tan^{-1}(-\infty) + \tan^{-1}\infty - \tan^{-1} a.$$

$$= \tan^{-1}\infty + \tan^{-1}\infty.$$

$$= \pi/2 + \pi/2$$

$$= \pi.$$

$\therefore$ Given integral is convergent.

Comparison Test

$\int_a^\infty f(x)\,dx$ be a improper integral of IInd kind.

If $f(x)$ and $g(x)$ be the two functions which are bounded and integrable and $g(x)$ is positive and $|f(x)| \leq g(x)$, then.

$\int_a^\infty f(x)\,dx$ convergent when $\int_a^\infty g(x)\,dx$ convergent.

Alternative form.

$\lim_{x\to\infty} \frac{f(x)}{g(x)}$ = a finite number.

Then, $\int_a^\infty f(x)\,dx$, $\int_a^\infty g(x)\,dx$ either convergent or divergent.

Theorem :- $\int_a^\infty \frac{dx}{x^n}$ convergent if $n>1$ and divergent if $n\leq$

Sol^n:- Case I.

$n>1$.

$$\int_a^\infty \frac{dx}{x^n} = \lim_{X\to\infty} \int_a^X \frac{dx}{x^n}.$$

$$= \lim_{X\to\infty} \left[\frac{x^{-n+1}}{1-n}\right]_a^X.$$

$$= \lim_{X\to\infty} \left[\frac{X^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right]$$

$$\cancel{= \lim_{X\to\infty} \left[\frac{1}{X^{n-1}(1-n)} - \frac{a^{1-n}}{1-n}\right]}$$

~~Of~~

if

$n>1$

$\Rightarrow 0 > 1-n$.

$$\therefore \int_a^\infty \frac{dx}{x^n} = 0 - \frac{a^{1-n}}{1-n}.$$

Then the given integral is convergent when $n>1$.

Case II

$n < 1$

$\Rightarrow 0 < 1-n.$

Then,
$$\int_a^\infty \frac{dx}{x^n} = \lim_{X\to\infty}\left[\frac{X^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right]$$

$$= \infty.$$

Then the given integral is divergent when $n<1$.

Case III

$n=1$

$$\int_a^\infty \frac{dx}{x^n} = \int_a^\infty \frac{dx}{x}$$

$$\cancel{=[\log x]} \quad = \lim_{X\to\infty}\int_a^X \frac{dx}{x}$$

$$= \lim_{X\to\infty}\left[\log x\right]_a^X$$

$$= \lim_{X\to\infty}\left[\log X - \log a\right]$$

$$= \infty.$$

$\therefore$ The given integral is divergent when $n=1$.

$\therefore$ Given Integral is Convergent when $n>1$. and divergent when $n\leq 1$.

Ex-16 Test the Convergency $\int_1^{\infty} \frac{dx}{x\sqrt{1+x^2}}$

Solⁿ :-

$$f(x) = \frac{1}{x\sqrt{1+x^2}}$$

Taking, $g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{x\sqrt{1+x^2}} \times x^2$$

$$= \lim_{x\to\infty} \frac{1}{x \cdot x\sqrt{\frac{1}{x^2}+1}} \cdot x^2$$

$$= \frac{1}{\sqrt{\frac{1}{\infty}+1}} = \frac{1}{\sqrt{0+1}} = 1, \text{ a finite no.}$$

$\therefore \int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{dx}{x^2}$ is convergent because $n=2$ greater than 1.

So, by comparison test $\int_1^{\infty} \frac{dx}{x\sqrt{1+x^2}}$ is convergent.

Ex-17 Test the Convergency of the integral $\int_1^{\infty} \frac{dx}{\sqrt{x^3+1}}$

Solⁿ :- $f(x) = \frac{1}{\sqrt{x^3+1}}$

Taking $g(x) = \frac{1}{x^{3/2}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{x^3+1}} \cdot x^{3/2}$$

$$= \lim_{x\to\infty} \frac{1}{x^{3/2}\sqrt{1+\frac{1}{x^3}}} \cdot x^{3/2}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^3}}}$$

$= 1$, a finite no.

$\therefore \int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{dx}{x^{3/2}}$ is Convergent because $n = \frac{3}{2}$ greater than 1.

Hence, by comparison test the given integral $\int_1^{\infty} \frac{dx}{\sqrt{x^3+1}}$ is convergent.

Ex-18 Test the convergency of $\int_{2}^{\infty} \frac{dx}{\sqrt{x^2-1}}$

Soln:-

$$f(x) = \frac{1}{\sqrt{x^2-1}}$$

Consider a f^n $g(x) = \frac{1}{x}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2-1}} \cdot x$$

$$= \lim_{x\to\infty} \frac{1}{x\sqrt{1-\frac{1}{x^2}}} \cdot x$$

$$= \frac{1}{\sqrt{1-0}} = 1, \text{ a finite number.}$$

$\therefore \int_{2}^{\infty} g(x)\,dx = \int_{2}^{\infty} \frac{1}{x}$ is divergent because $n = 1$.

Hence, by comparison test, given integral $\int_{2}^{\infty} \frac{dx}{\sqrt{x^2-1}}$ is divergent.

Ex-19 Test the convergency of the integral $\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\,dx$.

Soln:- $\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \int_{0}^{a} \frac{\sin^2 x}{x^2}\,dx + \int_{a}^{\infty} \frac{\sin^2 x}{x^2}\,dx$.

$$= I_1 + I_2.$$

$\frac{\sin x}{x} = 1$.

$\frac{e^x - 1}{x} = 1$

I_1 is proper integral. So, it is convergent.

Test for convergency of I_2

$$I_2 = \int_{a}^{\infty} \frac{\sin^2 x}{x^2}\,dx.$$

$$f(x) = \frac{\sin^2 x}{x^2}.$$

$$|f(x)| = \left|\frac{\sin^2 x}{x^2}\right| = \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}.$$

Taking, $g(x) = \frac{1}{x^2}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}$$

$\therefore \int_a^\infty g(x)\,dx = \int_a^\infty \frac{1}{x^2}\,dx$ is Convergent because $n=2$ greater than 1.

So, by comparison test I_2 is Convergent.

Since, I_1 and I_2 are both Convergent.

So, $I = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is also Convergent.

Ex-20 Test the Convergency of $\int_0^\infty e^{-x} \frac{\sin x}{x}\,dx$.

Solⁿ:- $I = \int_0^\infty e^{-x} \frac{\sin x}{x}\,dx$.

$$I = \int_0^a e^{-x} \frac{\sin x}{x}\,dx + \int_a^\infty e^{-x} \frac{\sin x}{x}\,dx$$

$$I = I_1 + I_2$$

$$I_1 = \int_0^a e^{-x} \frac{\sin x}{x}\,dx.$$

I_1 is a proper integral. So, it is Convergent.

$$I_2 = \int_a^\infty e^{-x} \frac{\sin x}{x}\,dx.$$

$$f(x) = e^{-x} \frac{\sin x}{x}$$

$$|f(x)| = \left|e^{-x} \frac{\sin x}{x}\right| = e^{-x}\left|\frac{\sin x}{x}\right| \le e^{-x}.$$

Taking $g(x) = e^{-x}$.

$$\int_a^\infty g(x) = \int_a^\infty e^{-x}\,dx.$$

$$= \lim_{X\to\infty} \int_a^X e^{-x}\,dx.$$

$\int_a^{\infty} g(x)dx = \lim_{x\to\infty} [\ldots]_a^x$

$= -\lim_{x\to\infty} [e^{-x} - e^{-a}]$

$= -[0 - e^{-a}]$

$= \frac{1}{e^a}$, a finite no.

Hence, $\int_a^{\infty} g(x)dx$ is Convergent.

So, by comparison test $\int_a^{\infty} f(x)dx$ is convergent. i.e. I_2 is convergent.

Since, both I_1 and I_2 is Convergent.

So, $I = \int_0^{\infty} e^{-x} \frac{\sin x}{x} dx$ is Convergent.

Ex-21 Test the Convergency $\int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx$.

Solⁿ:- $\int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx = \int_0^{a} \frac{\cos mx}{x^2+a^2} dx + \int_a^{\infty} \frac{\cos mx}{x^2+a^2} dx$.

$= I_1 + I_2$

$I_1 = \int_0^{a} \frac{\cos mx}{x^2+a^2} dx$.

I_1 is a proper integral. So, it is always Convergent.

$I_2 = \int_a^{\infty} \frac{\cos mx}{x^2+a^2} dx$.

$f(x) = \frac{\cos mx}{x^2+a^2}$

$|f(x)| = \left| \frac{\cos mx}{x^2+a^2} \right| \leq \frac{1}{x^2+a^2}$

Taking $g(x) = \frac{1}{x^2+a^2}$

$\int_a^{\infty} g(x)dx = \lim_{x\to\infty} \int_a^{x} \frac{1}{x^2+a^2}$

$\therefore \int_a^{\infty} g(x)\,dx = \lim_{X\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_a^{X}$

$= \frac{1}{a}\lim_{X\to\infty}\left[\tan^{-1}\frac{X}{a} - \tan^{-1}1\right]$

$= \frac{1}{a}\left[\tan^{-1}\infty - \tan^{-1}1\right]$

$= \frac{1}{a}\left[\pi/2 - \pi/4\right]$

$= \frac{1}{a}\cdot\pi/4$, a finite no.

So, $\int_a^{\infty} g(x)\,dx$ is Convergent.

So, by Comparison test, I_2 is Convergent.

Since, both I_1 and I_2 are Convergent.

$\therefore I = \int_0^{\infty} \frac{\cos mx}{x^2+a^2}\,dx$ is convergent.

Ex-22 Test the Convergency $\int_0^{\infty} e^{-x^2}\,dx$.

Soln:- $I = \int_0^{\infty} e^{-x^2}\,dx = \int_0^{1} e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx$

$= I_1 + I_2$.

$I_1 = \int_0^{1} e^{-x^2}\,dx$.

I_1 is a proper integral. So, it is Convergent.

$I_2 = \int_1^{\infty} e^{-x^2}\,dx$.

$f(x) = e^{-x^2}$

$|f(x)| = |e^{-x^2}| \le x e^{-x^2} \quad x > 1$.

Taking,

$$g(x) = x e^{-x^2}.$$

$$\int_1^\infty g(x)\,dx = \int_1^\infty x e^{-x^2} dx.$$

$$= \lim_{X\to\infty} \int_1^X x e^{-x^2} dx.$$

| $x^2 = t$
| $2x\,dx = dt$.

$$= \lim_{X\to\infty} \left[\left(-\frac{1}{2}\right) e^{-x^2}\right]_1^X$$

$$= -\frac{1}{2} \lim_{X\to\infty} \left[e^{-X^2} - e^{-1}\right].$$

$$= -\frac{1}{2} \lim_{X\to\infty} \left[\frac{1}{e^{X^2}} - \frac{1}{e}\right]$$

$$= -\frac{1}{2} \left[0 - \frac{1}{e}\right)$$

$$= \frac{1}{2e}.,$$ a finite no.

So, $g(x)$ is convergent.

So, by comparison test $\int_1^\infty f(x)$ is convergent i.e. I_2 is convergent.

Since, Both I_1 and I_2 are convergent.

$\therefore$ $I = \int_0^\infty e^{-x^2} dx$ is Convergent.

μ- test

$f(x)$ be a bounded and integrable (a,∞), $a>0$.

if there is a no $\mu > 1$ and $\lim_{x\to\infty} x^{\mu} f(x)$ exist

and non-zero then $\int_a^{\infty} f(x)\,dx$ is convergent.

if $\mu \le 1$ then $\int_a^{\infty} f(x)\,dx$ divergent.

also, $\lim_{x\to\infty} x^{\mu} f(x) = \infty$ or $-\infty$, then given

integration is divergent.

Ex-23 Test the convergency of $\int_1^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}$

Solⁿ:- $f(x) = \frac{1}{x^{1/3}(1+x^{1/2})}$

$$\mu = \frac{5}{6} - 0 = 5/6.$$

$$\lim_{x\to\infty} x^{\mu} f(x) = \lim_{x\to\infty} x^{5/6} \frac{1}{x^{1/3}(1+x^{1/2})}$$

$$= \lim_{x\to\infty} x^{5/6} \frac{1}{x^{1/3}.x^{1/2}(1/x^{1/2}+1)}$$

$$= \frac{1}{(0+1)} = 1, \text{ a finite no.}$$

$\mu = 5/6 < 1$. So, by μ test, The given integral

$\int_1^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}$ is divergent.

Ex-24 Test the convergency of $\int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)}$

Solⁿ

$$f(x) = \frac{1}{\sqrt{x}(1+x)}$$

$$\mu = \frac{3}{2} - 0 = 3/2.$$

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{\mu} \frac{1}{\sqrt{x}(1+x)}$$ (13)

$$= \lim_{x \to \infty} x^{3/2} \frac{1}{x^{1/2} \cdot x^2 \left(\frac{1+x}{x^2}\right)} = \lim_{x \to \infty} x^{3/2} \frac{1}{x^{1/2} \cdot x \left(\frac{1}{x}+1\right)}$$

$$= \lim_{x \to \infty} x^{3/2} \frac{1}{x^{1/2} \cdot x \left(\frac{1}{x}+1\right)}$$

$$= \lim_{x \to \infty} \left(\frac{1}{\frac{1}{x}+1} \right)$$

$$= \left(\frac{1}{0+1} \right) = 1, \text{ a finite no.}$$

So, by μ test, given integral is convergent since $\mu = 3/2$ greater than 1.

Ex-25 Test the convergency.

$$\int_a^{\infty} \frac{dx}{x(\log x)^{n+1}}$$

Soln:- Let, $\log x = t$.

$$\Rightarrow \frac{dt}{dx} = \frac{1}{x}.$$

$$\int_{\log a}^{\infty} \frac{dt}{t^{n+1}}$$

Take $\mu = n+1$.

$$\lim_{t \to \infty} t^{\mu} f(t) = \lim_{t \to \infty} t^{n+1} \frac{1}{t^{n+1}}$$

$$= 1.$$

So, given integral is convergent if

$n+1 > 1$.

$\Rightarrow n > 0$.

and divergent if $n+1 \le 1 \Rightarrow n \le 0$.

... show that $\int_1 x^{n-1} e^{-x}\, dx$ is ...

Solⁿ :- $f(x) = x^{n-1} e^{-x}$.

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{\mu} x^{n-1} e^{-x}$$

$$= \lim_{x \to \infty} x^{\mu+n-1} e^{-x}.$$

$$= \lim_{x \to \infty} \frac{x^{\mu+n-1}}{e^{x}}.$$

$= 0$ for all values of μ and n.

If we take $\mu > 1$, then given integral is convergent.

Hence Proved

Ex-27 Test the convergency of $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$.

Solⁿ :- $$\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx = \int_0^{a} \frac{x^{2m}}{1+x^{2n}}\, dx + \int_a^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx.$$

$$= I_1 + I_2 \quad \text{—— (1)}.$$

$$I_1 = \int_0^{a} \frac{x^{2m}}{1+x^{2n}}\, dx.$$

I_1 is a Proper integral. So, it is obviously convergent.

Test of I_2

$$I_2 = \int_a^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx.$$

$\mu = 2n - 2m$.

$$\lim_{x\to\alpha} x^{\mu} f(x) = \lim_{x\to\alpha} x^{2n-2m} \frac{x^{2m}}{1+x^{2n}}$$ (14)

$$= \lim_{x\to\alpha} x^{2n-2m} \frac{x^{2m}}{x^{2n}\left(\frac{1}{x^{2n}}+1\right)}$$

$$= \lim_{x\to\alpha} x^{2n-2m} \frac{1}{x^{2n-2m}\left(\frac{1}{x^{2n}}+1\right)}$$

$$= \frac{1}{(0+1)} = 1,$$ ~~a finite no.~~ limit exist

So, I_2 is Convergent if $\mu > 1$.

$\Rightarrow 2n - 2m > 1$. which is Possible only when

$n > m$.

I_2 is divergent if $\mu \leq 1$.

$\Rightarrow 2n - 2m \leq 1$.

$\Rightarrow n \leq m$.

So, Given integral is Convergent when $n > m$.

and divergent when $n \leq m$.

Abel's Test of Convergency.

If $\int_a^\infty f(x)dx$ be an improper integral and convergent and $\phi(x)$ is monotonic and bounded then by Abel's test

$$\int_a^\infty f(x)\,\phi(x)\,dx \text{ is convergent.}$$

Ex- 28 Test the convergency of $\int_a^\infty e^{-x} \frac{\sin x}{x^2} dx$.

Solⁿ:- Let, $f(x) = \frac{\sin x}{x^2}$

$\phi(x) = e^{-x}$.

$$|f(x)| = \left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$$

Taking $g(x) = \frac{1}{x^2}$

$\int_a^\infty g(x)dx = \int_a^\infty \frac{1}{x^2}dx$ is convergent because $n=2>1$

$\phi(x) = e^{-x}$ which is always bounded and monotonic decreasing.

So, By Abel's Test $\int_a^\infty f(x)\phi(x)dx$ is convergent.

$\therefore \int_a^\infty e^{-x} \frac{\sin x}{x^2} dx$ is convergent.

Ex-29 Test the Convergency of $\int_a^{\infty}(1-e^{-x})\frac{\cos x}{x^2}\,dx$.

Solⁿ:- Let, $f(x) = \frac{\cos x}{x^2}$ (15)

$\phi(x) = 1-e^{-x}$.

$|f(x)| = \left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$

Taking $g(x) = \frac{1}{x^2}$

$\int_a^{\infty} g(x)\,dx = \int_a^{\infty}\frac{dx}{x^2}$ is convergent because $n=2$ greater than 1.

So, by comparison test $\int_a^{\infty} f(x)\,dx$ is convergent.

$\phi(x) = 1-e^{-x}$, which is always bounded and monotonically increasing.

So, by Abel's Test $\int_a^{\infty} f(x)\phi(x)\,dx$ is convergent.

$\therefore \int_a^{\infty}(1-e^{-x})\frac{\cos x}{x^2}\,dx$ is convergent.

Dirichlet's Test

If $f(x)$ is bounded and monotonic on the interval $a \leq x < \infty$. $\lim_{x\to\infty} f(x) = 0$ then $\int_a^{\infty} f(x)\phi(x)\,dx$ convergent where $\left|\int_a^{x}\phi(x)\,dx\right|$ be bounded $\forall x$.

Ex-30 Test the Convergency $\int_a^{\infty}\frac{\sin x}{\sqrt{x}}\,dx$.

Solⁿ:- $f(x) = \frac{1}{\sqrt{x}}$. $\phi(x) = \sin x$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$.

Now, $\left|\int_a^{x}\phi(x)\,dx\right| = \left|\int_a^{x}\sin x\,dx\right| = \left|[-\cos x]_a^x\right| = \left|-(\cos x - \cos a)\right|$

$= |\cos a - \cos x| \leq 2$.

So, $\int_a^{x}\phi(x)\,dx$ is bounded.

So, by Dirichlet test $\int_a f(x)\phi(x)\,dx$ is convergent.

$\therefore \int_a^\infty \frac{\sin x}{\sqrt{x}}\,dx$ is convergent.

Ex-31 Test the Convergency $\int_0^\infty \frac{\sin x}{x}\,dx$.

Soln:- $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^a \frac{\sin x}{x}\,dx + \int_a^\infty \frac{\sin x}{x}\,dx.$

$= I_1 + I_2.$

Test for I_1

$I_1 = \int_0^a \frac{\sin x}{x}\,dx$

which is a proper integral.

So, it is obviously convergent.

Test for I_2

$I_2 = \int_a^\infty \frac{\sin x}{x}\,dx.$

Let, $f(x) = \frac{1}{x}$, $\phi(x) = \sin x$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0.$

Now,

$$\left|\int_a^x \phi(x)\,dx\right| = \left|\int_a^x \sin x\,dx\right|$$

$$= \left|-[\cos x]_a^x\right|$$

$$= |\cos a - \cos x| \le 2.$$

So, $\int_a^x \phi(x)\,dx$ is bounded

$\therefore$ By Dirichlet test $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

i.e. $\int_a^\infty \frac{\sin x}{x}\,dx$ is convergent.

So, I_2 is convergent.

Since, I_1 and I_2 are convergent.

$\therefore$ I is also convergent

$\therefore \int_0^\infty \frac{\sin x}{x} dx$ is Convergent.

Ex-32 Test the Convergency of $\int_0^\infty e^{-x} \frac{\sin x}{x} dx$.

Ans:- $\int_0^\infty e^{-x} \frac{\sin x}{x} dx = \int_0^a e^{-x} \frac{\sin x}{x} dx + \int_a^\infty e^{-x} \frac{\sin x}{x} dx$.

$= I_1 + I_2$.

Test of I_1

$I_1 = \int_0^a e^{-x} \frac{\sin x}{x} dx$., is a proper integral.

So, it is obviously Convergent.

Test of I_2

$I_2 = \int_a^\infty e^{-x} \frac{\sin x}{x} dx$.

$f(x) = \frac{e^{-x}}{x}$, $\phi(x) = \sin x$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{-x}}{x} = 0$.

Now,

$\left|\int_a^x \phi(x) dx\right| = \left|\int_a^x \sin x \, dx\right|$.

$= \left|[-\cos x]_a^x\right|$

$= |\cos a - \cos x| \leq 2$

So, $\int_a^x \phi(x)$ is bounded.

$\therefore$ By Dirichlet test $\int_a^\infty f(x)dx$ is Convergent i.e. I_2 is Convergent

$\therefore$ Both I_1 and I_2 are Convergent. So, I is also Convergent.

$\therefore \int_0^\infty e^{-x} \frac{\sin x}{x} dx$ is Convergent.

Ex-33 Show that $\int_1^{\infty} \frac{\sin x}{x^p}$ converges ...

Solⁿ:-

$$\left|\frac{\sin x}{x^p}\right| = \frac{|\sin x|}{x^p} \le \frac{1}{x^p}.$$

Taking, $g(x) = \frac{1}{x^p}$.

$$\int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{dx}{x^p} \text{ Convergent because } p>1.$$

So, $\int_1^{\infty} \left|\frac{\sin x}{x^p}\right| dx$ is convergent.

Hence, $\int_1^{\infty} \frac{\sin x}{x^p} dx$ ~~abs~~ converges absolutely if $p>1$.

Ex-34. Test the convergency of $\int_0 e^{-mx} x^n\,dx$.

Solⁿ:- $f(x) = e^{-mx} x^n$.

$n \geq 0$

$m > 0$.

Then given integral is proper integral.

Then it is obviously convergent.

Now,

let, $n < 0$.

$$\lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{\mu} e^{-mx} x^n.$$

$$= \lim_{x\to 0} x^{\mu+n} e^{-mx}$$

$=$ limit exist if $\mu+n=0$.

$\Rightarrow \mu = -n$.

So, by μ test.

Given integral is convergent for $0<\mu<1$

$\Rightarrow 0 < -n < 1$.

$\Rightarrow 0 > n > -1$.

Hence given integral is convergent for $-1 < n$.