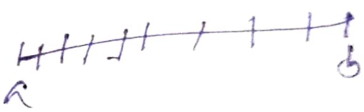
C-4.2

Partition of a closed interval:

Suppose $I=[a,b]$ be a closed and bounded interval then by partition of I means a finite sets of real numbers $P=\{x_1, x_2, \ldots, x_n\}$ Having the property that $a=x_0 \le x_1 \le x_2 \le \cdots \le x_n=b$



Note:

1) $I_1=[x_0, x_1]$, $I_2=[x_1, x_2]$, $I_3=[x_2, x_3]$ .....
$I_i=[x_{i-1}, x_i]$, ... $I_n=[x_{n-1}, x_n]$ ~~this~~ are the subinterval of $[a,b]$.

ii) We shall used the symbol $\Delta x_i = x_i - x_{i-1}$ to denote the length of i^{th} sub-interval.

iii) For any partition P the length of the largest sub-interval is called norm or mesh of the partition and is denoted by $\mu(P)$.

Riemann sum

Suppose $f(x)$ is bounded function defined in a closed interval $[a, b]$ and suppose $P = \{x_0, x_1, x_2, \dots x_n\}$ is a partition of $[a, b]$, let.

$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$, $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$.

Therefore

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + m_n \Delta x_n$$

is called the Lower R-sum of $f(x)$ on $[a, b]$ with respect to P.

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + \dots + M_n \Delta x_n$$

is called the upper Riemann sum of $f(x)$ on $[a, b]$ with respect to partition P.

~~R~~. Darboux condition of Riemann integrability

When two integrals

$\overline{\int} f dx$ or $\overline{\int_a^b} f(x)\,dx = \inf \{U(P,f) : P \text{ is a partition of } [a,b]\}$

~~upper~~ integral as and

$\underline{\int} f dx$ or $\underline{\int_a^b} f(x)\,dx = \sup \{L(P,f) : P \text{ is a partition of } [a,b]\}$.

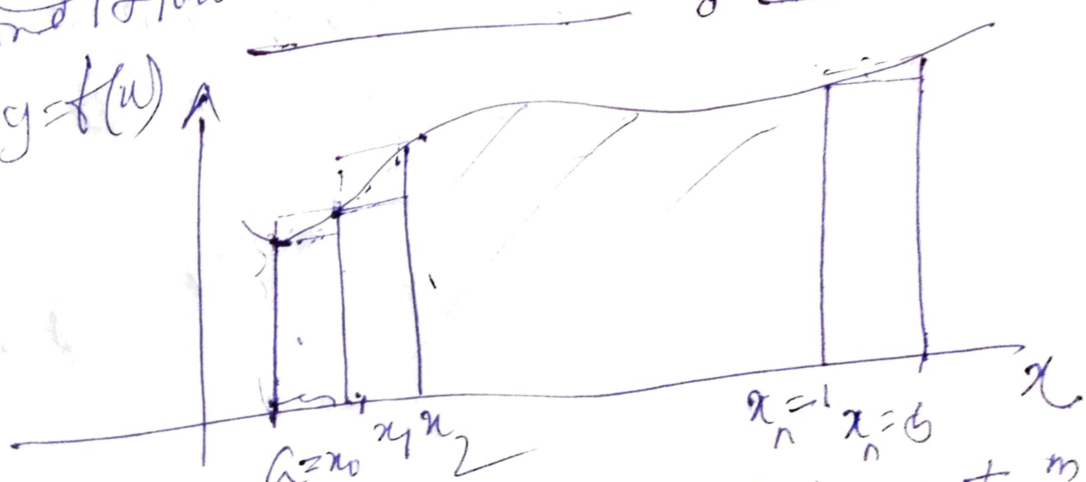
lower ~~upper~~ integral

are equal then that condition i.e

$$\int_a^b f dx = \underline{\int_a^b} f\,dx = \overline{\int_a^b} f dx$$

is known as

Riemann integrability condition.



$L(P,f) = \sum m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$

$U(P,f) = \sum M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$

Q Show that a constant function K is integrable and $\int_a^b K\,dx = K(b-a)$

$m_1\ M_1\ m_2\ M_2 \quad f(x)$

$x_0\ x_1\ x_2 \ \dots\ x_n$

$a \ \dots\ b$

Sol^n: For any partition P of the interval $[a,b]$,

we have $L(P,f) = \sum m_i \Delta x_i = m_1\Delta x_1 + m_2\Delta x_2 + \dots + m_n\Delta x_n$

$$L(P,f) = K\Delta x_1 + K\Delta x_2 + \dots + K\Delta x_n$$

$$= K(\Delta x_1 + \Delta x_2 + \dots + \Delta x_n) = K(x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \dots + x_n - x_{n-1})$$

$$= K(x_n - x_0) = K(b-a)$$

$$\Rightarrow \int_a^b K\,dx = \sup L(P,f) = K(b-a)$$

Now $U(P,f) = \sum M_i \Delta x_i = M_1\Delta x_1 + M_2\Delta x_2 + \dots + M_n\Delta x_n = K\Delta x_1 + K\Delta x_2 + \dots + K\Delta x_n = K(b-a)$

$$\int_a^b K\,dx = \inf U(P,f)$$

$$\cancel{= \inf(K\Delta x_1 + K\Delta x_2 + \dots + K\Delta x_n)}$$

$$= K(b-a)$$

Thus $\int_a^b K\,dx = \int_a^b K\,dx = K(b-a)$

Which implies that the function K is integrable and

$$\int_a^b K\,dx = K(b-a)$$

$f(x) = x^2$, M

$= 1, 2, 1$

$[1,2]$ M_2

$m_1 = 1$, $M_1 = 1.21$, 1.44

$[1, 1.1]$ $[1.1, 1.2]$ $\dots$ $[1.9, 2]$

a) Show that the function f defined by

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$$

is not integrable on any interval

Soln Let,
For any partition $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ of the interval $[a, b]$ we have,

$$L(P,f) = \sum m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

$$L(P,f) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

$$= 0 \Delta x_1 + 0 \Delta x_2 + \cdots + 0 \Delta x_n$$

$$= 0$$

$$\underline{\int} f\,dx = \sup L(P,f) = 0$$

Now, $U(P,f) = \sum M_i \Delta x_i$

$$= M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$$

$$= 1 \Delta x_1 + 1 \Delta x_2 + \cdots + 1 \Delta x_n$$

$$= b - a$$

$\overline{\int} f dx = \inf U(P,f) = b-a$

Thus, $\underline{\int} f dx \neq \overline{\int} f dx$

which implies that the function is not integrable on any ~~closed~~ closed interval ~~[a,b]~~ [a,b]

Inequalities of upper and lower sums

Q) Prove that for any function $L(P,f) \leq U(P,f)$ on an interval $[a,b]$.

Proof: Given,

$f(x)$ is bounded on $[a,b]$

Let, $m_i = \inf\{f(x) : f(x) \in [x_{i-1}, x_i]\}$

and $M_i = \sup\{f(x) : f(x) \in [x_{i-1}, x_i]\}$.

clearly. We know that,

$$m_i \leq M_i$$

$$\Rightarrow m_i \Delta x_i \leq M_i \Delta x_i$$

$$\Rightarrow \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i$$

$$\Rightarrow L(P,f) \leq U(P,f)$$

Proved.

Refinement of ~~a~~ partition

Norm: For any partition P $M(P)$ is called norm of the partition if $M(P) = \max \Delta x_i$.

Refinement: A partition P^* is said to be a refinement of P if $P^* \supseteq P$.

Theorem: If P^* is a refinement of a partition P, then for a bounded function f,

1) $L(P^*, f) \geq L(P, f)$, and

ii) $U(P^*, f) \leq U(P, f)$.

Soln: To prove ①, suppose first that P^* contains just one point more than P.

Let this extra point be ξ, and suppose that this point is in Δx_i, that is, $x_{i-1} < \xi < x_i$

As the function is bounded over the

entire interval $[a,b]$, it is bounded in every sub-interval $\Delta x_i (i = 1, 2, \dots, n)$. Let w_1, w_2, m_i be the infimum (g.l.b) of f in the intervals $[x_{i-1}, \xi]$, $[\xi, x_i]$, $[x_{i-1}, x_i]$ respectively.

Note:

$P^* = \{ \overset{a}{x_0}, x_1, \dots x_{i-1}, \xi, x_i, \dots \overset{=b}{x_n} \}$

$P = \{ \overset{a}{x_0}, x_1, \dots x_i, \dots \overset{=b}{x_n} \}$

Clearly $m_i \le w_1$, $m_i \le w_2$.

$\therefore L(P^*, f) - L(P, f)$

$= \cancel{m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + m_i \Delta x_i + \dots + m_n \Delta x_n}$

$= \{ m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + w_1 (\xi - x_{i-1}) + w_2 (x_i - \xi) + \dots + m_n \Delta x_n \} - \{ m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + m_i \Delta x_i + \dots m_n \Delta x_n \}$

$= \cancel{m_1 \Delta x_1}$

$= w_1 (\xi - x_{i-1}) + w_2 (x_i - \xi) - m_i (x_i - x_{i-1})$

$= w_1 (\xi - x_{i-1}) - m_i (\xi - x_{i-1}) + m_i (\xi - x_{i-1}) + w_2 (x_i - \xi) - m_i (x_i - x_{i-1})$

$= (w_1 - m_i)(\xi - x_{i-1}) + m_i \xi - \cancel{m_i x_{i-1}} + w_2 x_i - w_2 \xi - m_i x_i + \cancel{m_i x_{i-1}}$

$= (w_1 - m_i)(\xi - x_{i-1}) + m_i (\xi - x_i) - w_2 (\xi - x_i)$

$= (w_1 - m_i)(\xi - x_{i-1}) + (\xi - x_i)(m_i - w_2) \ge 0$

$= (w_1 - m_i)(\xi - x_{i-1}) + (x_i - \xi)(w_2 - m_i) \geq 0$.

Q) If f is defined on $[0,a]; a>0$ by $f(x)=x^2, \forall x \in [0,a]$ then $f \in R[0,a]$ & $\int_0^a f(x)\,dx = \frac{a^3}{3}$

Sol^n:

Given, $f(x) = x^2, \forall x \in [0,a]$ For

Let, P be a partition of $[0,a]$

$\therefore P : \{0, 0+\frac{a}{n}, 0+2\frac{a}{n}, \ldots, \cancel{0+n\frac{a}{n}}\ 0+\frac{(i-1)a}{n}, 0+\frac{ia}{n}, \ldots, 0+n\frac{a}{n}\}$

$\left\{ h = \frac{\text{end pt} - \text{initial pt}}{n} = \frac{a-0}{n} = \frac{a}{n} \right.$

$P : \{0, 0+\frac{a}{n}, 0+2\frac{a}{n}, \ldots, 0+\frac{ia}{n}, \ldots, a\}$

i.e. $P : \{0, \frac{a}{n}, \frac{2a}{n}, \ldots \frac{ia}{n}, \ldots, a\}$

Now, $L(P,f) = \sum m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \ldots + m_n \Delta x_n$

$\cancel{= 0\left(\frac{a}{n}\right) + \frac{a}{n}\left(\frac{a}{n}\right) + \ldots + 0\left(\frac{a}{n}\right)}$

$\cancel{= 0 + \left(\frac{a}{n}\right)^2}$

$= f(0)\Delta x_1 + f\left(\frac{a}{n}\right)\Delta x_2 + \ldots f\left(\frac{(n-1)a}{n}\right)^2 \Delta x_n$

$= 0^2 \cdot \frac{a}{n} + \left(\frac{a}{n}\right)^2\left(\frac{a}{n}\right) + \ldots + \left(\frac{(n-1)a}{n}\right)^2 \frac{a}{n}$

$= 0 + \left(\frac{a}{n}\right)^2 + 2^2\left(\frac{a}{n}\right)^3 + 3^2\left(\frac{a}{n}\right)^3 + \ldots + (n-1)\left(\frac{a}{n}\right)^3$

$$= \frac{a^3}{n^3}\left\{1^2+2^2+3^2+\cdots+(n-1)^2\right\}$$

$$= \left(\frac{a}{n}\right)^3 \frac{n(n-1)(2n-1)}{6}$$

Note:

$$1^2+2^2+3^2+\cdots+(n-1)^2 = \frac{n(n-1)(2n-1)}{6}$$

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Again,

$$U(P,f) = \sum M_i \Delta x_i$$

$$= M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + \cdots + M_n \Delta x_n$$

$$= \left(\frac{a}{n}\right)^2\left(\frac{a}{n}\right) + \left(\frac{2a}{n}\right)^2\left(\frac{a}{n}\right) + \left(\frac{3a}{n}\right)^2\left(\frac{a}{n}\right) + \cdots + (a)^2\left(\frac{a}{n}\right)$$

$$= \left(\frac{a}{n}\right)^3 + 2^2\left(\frac{a}{n}\right)^3 + 3^2\left(\frac{a}{n}\right)^3 + \cdots + n^2\left(\frac{a}{n}\right)^3$$

$$= \left(\frac{a}{n}\right)^3\left\{1^2+2^2+3^2+\cdots+n^2\right\}$$

$$= \left(\frac{a}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$$

Now $\int f\,dx = \sup\{L(P,f)\}$, P is any partitions of $[0,a]$

$$= \lim_{n\to\infty} \frac{a^3}{6}\,\frac{n(n-1)(2n-1)}{n^3}$$

$$= \frac{a^3}{6}\lim_{n\to\infty}\frac{n\cdot n\cdot n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{n^3}$$

$$= \frac{a^3}{6}(1-0)(2-0)$$

$$= \frac{a^3}{3}$$

Again,

$\overline{\int} f\,dx = \inf\{U(P,f)\}$, P is any partition of $[0,a]$

$$= \lim_{n\to\infty} \frac{a^3}{6}\,\frac{n(n+1)(2n+1)}{n^3}$$

$$= \frac{a^3}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{a^3}{6}(1+0)(2+0)$$

$$= \frac{a^3}{3}$$

$$\therefore \underline{\int} f\,dx = \overline{\int} f\,dx = \frac{a^3}{3}$$

Hence, $f \in R[a,b]$

and $\displaystyle\int_0^a f(x)\,dx = \frac{a^3}{3}$

Q) Let, $f(x) = x^3 - 1$, $x \in [0,1]$ show that, $f(x)$ is Riemann integral and find ~~integ~~ $\int_0^1 f(x)dx$

Sol^n: Given,

$$f(x) = x^3 - 1, \quad x \in [0,1]$$

Let, P be any partition on $[0,1]$

$$P : \left\{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{i-1}{n}, \frac{i}{n}, \dots \frac{n}{n} = 1\right\} \quad \left| h = \frac{1-0}{n}\right.$$

$$L(P,f) = \sum m_i \Delta x_i$$

$$= m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + m_n \Delta x_n$$

$$= f(0)\Delta x_1 + f\left(\frac{1}{n}\right)\Delta x_2 + \dots + f\left(\frac{n}{n}\right)\Delta x_n$$

$$= f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + \dots + f\left(\frac{n}{n}\right)\frac{1}{n}$$

$$= -1\frac{1}{n} + \frac{1-n^3}{n^3}\left(\frac{1}{n}\right) + \dots + 0\frac{1}{n}$$

$$= \frac{1}{n}\left\{-1 + \frac{1-n^3}{n^3} + \dots + 0\right\}$$

$$= -1\left(\frac{1}{n}\right) + \left\{\left(\frac{1}{n}\right)^3 - 1\right\}\frac{1}{n} + \left(\frac{2^3}{n^3} - 1\right)\frac{1}{n} + \dots + \left(\frac{n^3}{n^3} - 1\right)\frac{1}{n}$$

$$= \frac{1}{n}\left\{-1 + \left(\frac{1}{n^3} - 1\right) + \left(\frac{2^3}{n^3} - 1\right) + \dots + \left(\frac{n^3}{n^3} - 1\right)\right\}$$

$$= \frac{1}{n}\left\{(-1)n + \frac{1}{n^3} + \frac{2^3}{n^3} + \dots + \frac{n^3}{n^3}\right\}$$

$$= \frac{1}{n}\left[(-1)n + \frac{1}{n^3}\left\{1^3 + 2^3 + 3^3 + \dots + n^3\right\}\right]$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$+1 \times -1 = (-1)^2$$

$$-1 + -1 = -2$$

$$= \frac{1}{n}\left[(-1)n + \frac{1}{n^3}\left\{\left[\frac{n(n+1)}{2}\right]^2\right\}\right]$$

$$= \frac{1}{n}\left[(-1)n + \frac{1}{n^3}\left[\frac{n(n+1)}{2}\right]^2\right]$$

$$= -1 + \frac{1}{n^3} \cdot \frac{n^2(n+1)^2}{4} = -1 + \frac{(n+1)^2}{4n^2}$$

$$U(P,f) = \sum M_i \Delta x_i$$

$$= M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + \dots + M_n \Delta x_n$$

$$= \left(\frac{1^3}{n^3} - 1\right)\frac{1}{n} + \left(\frac{2^3}{n^3} - 1\right)\frac{1}{n} + \left(\frac{3^3}{n^3} - 1\right)\frac{1}{n} + \dots + \left(\frac{(n-1)^3}{n^3} - 1\right)\frac{1}{n}$$

$$= \frac{1}{n}\left[(-1)n + \frac{1}{n^3}\left\{1^3 + 2^3 + 3^3 + \dots (n-1)^3\right\}\right]$$

$$= \frac{1}{n}\left[(-1)n + \frac{1}{n^3}\left[\frac{n(n-1)}{2}\right]^2\right]$$

$$= -1 + \frac{(n-1)^2}{4n^2}$$

Now,

$$\int_0^1 f(x)\,dx = \sup L(P,f)$$

$$= \lim_{n \to \infty}\left(-1 + \frac{(n+1)^2}{4n^2}\right)$$

$$= \lim_{n \to \infty}\left(-1 + \frac{n^2\left(1 + \frac{1}{n}\right)^2}{4n^2}\right)$$

$3x+4, \ [2,3]$

$= -1 + \frac{1}{4}$

$= -\frac{3}{4}$

$$\overline{\int_0^1} f(x)\,dx = \inf U(P,f)$$

$$= \lim_{n\to\infty}\left[-1+\frac{(n-1)^2}{4n^2}\right]$$

$$= -1 + \lim_{n\to\infty}\frac{n^2\left(1-\frac{1}{n}\right)^2}{4n^2}$$

$$= -1 + \frac{1}{4}$$

$$= -\frac{3}{4}$$

$$\therefore \underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \int_0^1 f(x)\,dx = -\frac{3}{4}$$

Theorem Let $f(x)$ be bounded function defined on $[a,b]$ and let M and m be the supremum and infimum of $f(x)$ in $[a,b]$. Then for any partition P of $[a,b]$ we have

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

Proof: Let, $P = \{x_0, x_1, x_2, \dots, x_n\}$ and let M_i and m_i be the supremum and infimum of $f(x)$ in $[x_{i-1}, x_i]$. Then we have

$$m \le m_i \le M_i \le M$$

$$\therefore \sum_{i=1}^{n} m \Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \sum_{i=1}^{n} M \Delta x_i \to (1)$$

Since, $\sum_{i=1}^{n} m \Delta x_i = m \sum_{i=1}^{n} (x_i - x_{i-1})$

$$= m(x_1 - x_0 + x_2 - x_1 + \dots x_n - x_{n-1})$$

$$= m(x_n - x_0)$$

$$= m(b-a)$$

Similarly,

$$\sum_{i=1}^{n} M_i \Delta x_i = M(b-a)$$

$$(i) \Rightarrow m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

Hence proved

Theorem

Prove that lower R-integral cannot exceed the upper R-integral i.e

$$\underline{\int_a^b} f(x) \leq \overline{\int_a^b} f(x)$$

Proof: Suppose, P_1 and P_2 be any two partition of $[a,b]$ then

$$L(P_1,f) \leq U(P_2,f) \longrightarrow (1)$$

Now, keeping P_2 fixed and taking supremum overall P_1 we get

$$\underline{\int_a^b} f \leq U(P_2,f) \longrightarrow (2)$$

Again, keeping P_1 fixed and taking infimum overall P_2 we get,

$$\overline{\int_a^b} f \geq L(P_1,f) \longrightarrow (3)$$

From (2) and (3)

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Hence proved

Note

If $\int_a^b f \neq \int_a^{\bar{b}} f$, then we say that f is not Riemann integrable over [a, b]. If $f \in R[a,b]$ and $b < a$, Then we define

$$\int_a^b f\,dx = -\int_a^b f\,dx \text{ and then } \int_a^b f\,dx = 0$$

Refinement of a partition

Riemann Integrability

Let f be the bounded function defined on bounded [a, b]. Then f is Riemann integrable on [a, b] iff $\int_a^b f = \int_a^{\bar{b}} f$.

Their common value is known as Riemann integral of f on [a, b] and is denoted by $\int_a^b f$ or $\int_a^b f\,dx$. The set of all Riemann integral function is denoted by R. Thus $f \in R[a,b]$ means f is Riemann integral function over [a, b].

Q. If $f(x) = x^2$ on $[0,a]$ show that $f \in R[0,a]$ and find $\int_0^a f$.

Solⁿ: Given,

$$f(x) = x^2, \forall x \in [0,a]$$

Let, P be a partition of $[0,a]$

$$\therefore P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \dots \left(\frac{n-1}{n}\right)a, \frac{na}{n} = a\right\}$$

Here, $m_i = \frac{(r-1)^2}{n^2} a^2$, $M_i = \frac{a^2 r^2}{n^2}$

$\Delta x_i = \frac{a}{n}$ where, $i = 1, 2, 3 \dots n$

$$\therefore L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$= m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + \dots + m_n \Delta x_n$$

$$= f(0)\Delta x_1 + f\left(\frac{a}{n}\right)\Delta x_2 + f\left(\frac{2a}{n}\right)\Delta x_3 + \dots + f\left(\frac{(n-1)a}{n}\right)\Delta x_n$$

$$= 0 \cdot \frac{a}{n} + \left(\frac{a}{n}\right)^2 \frac{a}{n} + \left(\frac{2a}{n}\right)^2 \frac{a}{n} + \dots + \left(\frac{(n-1)a}{n}\right)^2 \frac{a}{n}$$

$$= \left(\frac{a}{n}\right)^3 + 2^2\left(\frac{a}{n}\right)^3 + \dots + (n-1)^2\left(\frac{a}{n}\right)^3$$

$$= \left(\frac{a}{n}\right)^3 \left\{1^2 + 2^2 + \dots + (n-1)^2\right\}$$

$$= \left(\frac{a}{n}\right)^3 \frac{n(n-1)(2n-1)}{6}$$

$$= a^3 \frac{(n-1)(2n-1)}{6n^2}$$

and,
$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$= f\left(\frac{a}{n}\right)\Delta x_1 + f\left(\frac{2a}{n}\right)\Delta x_2 + \dots + f\left(\frac{na}{n}\right)\Delta x_n$$

$$= \left(\frac{a}{n}\right)^2 \frac{a}{n} + \left(\frac{2a}{n}\right)^2 \frac{a}{n} + \dots + \left(\frac{na}{n}\right)^2 \frac{a}{n}$$

$$= \left(\frac{a}{n}\right)^3 \left\{1^2 + 2^2 + \dots n^2\right\}$$

$$= \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= a^3 \frac{(n+1)(2n+1)}{6n^2}$$

Now,
$$\underline{\int} f\,dx = \sup\{L(P,f)\}, \text{ P is any partition of } [0,a]$$

$$= \lim_{n\to\infty} a^3 \frac{(n-1)(2n-1)}{6n^2}$$

$$= \frac{a^3}{6} \lim_{n\to\infty} \frac{nn\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{n^2}$$

$$= \frac{a^3}{6}(1-0)(2-0)$$

$$= \frac{a^3}{3}$$

Notes given by Rajib sir