Limit of a sequence

A sequence $\{S_n\}$ has a limit at l if $\forall$ given $\epsilon > 0$ ~~there~~ $\exists$ a positive integer m such that $|S_n - l| < \epsilon \quad \forall \ n \geq m$.

$$\text{or} \quad -\epsilon < S_n - l < \epsilon$$

$$\Rightarrow l - \epsilon < S_n < l + \epsilon$$

$$\Rightarrow S_n \in (l - \epsilon, l + \epsilon) \quad \forall \ n \geq m$$

Note:- limit of a sequence is unique.

Limit point of a sequence:-

Let, $\{S_n\}$ be a sequence has a limit point ξ if $\forall$ given $\epsilon > 0$

$$|S_n - \xi| < \epsilon$$

(for some infinite numbers of n)

$$\Rightarrow -\epsilon < S_n - \xi < \epsilon$$

$$\Rightarrow \xi - \epsilon < S_n < \xi + \epsilon$$

$$\Rightarrow S_n \in (\xi - \epsilon, \xi + \epsilon),$$ for infinitely many n's.

Note:- limit point is not always unique.

$s_1 = -1,\ s_2 = 1,\ s_3 = -1, \ldots$

* Range $\{s_n\} = \{-1, 1\}$.

Let $\xi = 1$,

$s_n \in (1-\epsilon, 1+\epsilon)$ for infinitely many 'n'

$\therefore \xi = 1$ is a limit point of $\{s_n\}$.

Let $\xi = -1$,

$s_n \in (-1-\epsilon, -1+\epsilon)$ for infinitely many 'n'

$\therefore \xi = -1$ is also a limit point of $\{s_n\}$

$\therefore$ set of limit pts of $\{s_n\}$ is $\{-1, 1\}$.

*) **Limit inferior (lower limit)**

The smallest limit point of a sequence $\{s_n\}$ is called the limit inferior of $\{s_n\}$ and is denoted by $\liminf\limits_{n\to\infty} s_n$ or $\varliminf\limits_{n\to\infty} s_n$

Limit superior (upper limit)

The ~~highest li~~ greatest limit point of a sequence $\{s_n\}$ is called the limit superior of $\{s_n\}$ and is denoted by $\limsup\limits_{n\to\infty} s_n$ or $\varlimsup\limits_{n\to\infty} s_n$

C-4.2

Rajib sir

Note (i). Since the smallest limit point of a sequence $\langle S_n \rangle \leq$ the greatest limit point of $\langle S_n \rangle$, we have $\lim_{n\to\infty} \inf S_n \leq \lim_{n\to\infty} \sup S_n$

Note (ii) If $\langle S_n \rangle$ is not bounded above, we write $\lim_{n\to\infty} \sup S_n = \infty$

If $\langle S_n \rangle$ is not bounded below, we write $\lim_{n\to\infty} \inf S_n = -\infty$

Q. Find limit superior and inferior of $\langle S_n \rangle = \frac{1}{n}$, $n \in \mathbb{N}$

It has exactly one limit point, namely the set $\{0\}$ of limit point is bounded.

Sol^n: $$\lim_{n\to\infty} \inf S_n = 0 = \lim_{n\to\infty} \sup S_n$$

Let $l = 0$ and for every $\epsilon > 0$,

$S_n \in (0-\epsilon, 0+\epsilon)$, ~~man~~ for infinitely many n.

or $S_n \in (-\epsilon, \epsilon)$

$\therefore l = 0$ is a limit point

Set of limit points of $\langle S_n \rangle$ is $\{0\}$

$$\lim_{n\to\infty} \inf S_n = 0$$

$$\lim_{n\to\infty} \sup S_n = 0$$

Theorem:- A bounded sequence $\langle S_n \rangle$ converges to l if and only if

$$\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} \inf S_n = l.$$

Theorem If $\langle a_n \rangle$ and $\langle b_n \rangle$ are bounded sequences such that $a_n \leq b_n \ \forall \ n \in \mathbb{N}$, then

(i) $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$

(ii) $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$

Theorem If $\langle a_n \rangle$ and $\langle b_n \rangle$ are bounded sequences then

(i) $\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

(ii) $\liminf_{n\to\infty} (a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.

Q. Find the limit superior and limit inferior of each of the following sequences.

(i) $\langle 1, 2, 1, 2, \ldots \rangle$

Sol: Here, the ~~set of~~ limit points is $\{1, 2\}$ which is bounded.

$\therefore$ ~~$\limsup_{n\to\infty} S_n$~~

$\underline{\lim}_{n\to\infty} S_n = 1.$

$\overline{\lim}_{n\to\infty} S_n = 2.$

1 is ~~reap~~ repeated infinitely many times in this sequence, so it is a limit point. Similarly 2 is also a limit point

(ii) $\langle 1, 3, 5, 1, 3, 5, \ldots \rangle$

Sol: Here, the set of limit point is $\{1, 3, 5\}$ which is bounded.

$\therefore \underline{\lim\limits_{n\to\infty}} S_n = 1$

$\overline{\lim\limits_{n\to\infty}} S_n = 5$

(iii) $\left\langle 1 + \frac{(-1)^n}{n} \right\rangle = \langle S_n \rangle$

Sol: $S_n = \begin{cases} 1 - 1/n, & n \text{ is odd} \\ 1 + 1/n, & n \text{ is even} \end{cases}$

Here, $S_n \longrightarrow 1$ as $n \to \infty$

So, S_n has exactly one limit point i.e $\{1\}$ which is bounded.

$\therefore \overline{\lim\limits_{n\to\infty}} S_n = 1 = \underline{\lim\limits_{n\to\infty}} S_n.$

(iv) $\left\langle \frac{(-1)^n}{n^2} \right\rangle = \langle S_n \rangle$

Sol: $S_n = \begin{cases} \frac{-1}{n^2}, & n \text{ is odd} \\ \frac{1}{n^2}, & n \text{ is even} \end{cases}$

Here, $S_n \to 0$ as $n \to \infty$

So, S_n has exactly one limit point i.e $\{0\}$ which is bounded.

$\therefore \overline{\lim\limits_{n\to\infty}} S_n = 0 = \underline{\lim\limits_{n\to\infty}} S_n$

(v). $\langle (-1)^n (1+1/n) \rangle$

Sol: Let $\langle s_n \rangle = \langle (-1)^n (1+1/n) \rangle$

Now, $\langle s_n \rangle = \begin{cases} (-1)(1+1/n), & n \text{ is odd} \\ (1)(1+1/n), & n \text{ is even} \end{cases}$

Here, the set of limit points is $\{-1, 1\}$, which is bounded.

$\underline{\lim}_{n\to\infty} s_n = -1$

$\overline{\lim}_{n\to\infty} s_n = 1.$

(vi) $\langle (-1)^n n \rangle$

Theorem:-

Cauchy - Hadamard theorem:-

For every power series $\sum_0^\infty a_n z^n$, there exists a real number R, $0 \leq R < \infty$, called the ~~radius~~ radius of convergence, with the following properties:—

(I) The series converges absolutely for every $|z| < R$.

(II) The series diverges for $|z| > R$

(III) The series converges uniformly for $|z| \leq \rho$, where $0 \leq \rho < R$.

Proof:(I) We consider R according to the formula ~~$\frac{1}{R} = \overline{\lim_{n\to\infty}}$~~

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}.$$

Let $|z| < R$. Then there exists a real number ρ such that $|z| < \rho < R$.

So $\frac{1}{\rho} > \frac{1}{R}$

From definition of limit superior for every $\epsilon > 0$, there exists a (+)ve integer m such that

$$|a_n|^{1/n} < \frac{1}{R} + \epsilon \ \forall \ n \geq m$$

$$\Rightarrow |a_n|^{1/n} < \frac{1}{\rho} \ \forall \ n \geq m \quad \left[\frac{1}{\rho} > \frac{1}{R} \Rightarrow \frac{1}{\rho} = \frac{1}{R} + \epsilon\right]$$

$$\Rightarrow |a_n| < \frac{1}{\rho^n}, \ \forall \ n \geq m$$

$$\Rightarrow |a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \text{ for large values of } n.$$

Here the $\sum \left(\frac{|z|}{\rho}\right)^n$ is a geometric series with common ratio $\frac{|z|}{\rho} < 1$, since $|z| < \rho$ and so it converges. Therefore by comparison Test, $\sum |a_n z^n|$ is convergent and hence $\sum a_n z^n$ is absolutely convergent for $|z| < R$.

(ii) For $|z| > R$, we can choose a real number ρ ~~sch~~ such that $R < \rho < |z|$. So $\frac{1}{R} > 1/\rho$.

Again, since $\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$ for every $\epsilon > 0$, there exists a (+)ve integer m such that

$$|a_n|^{1/n} > 1/R - \epsilon, \ \forall n \geq m$$

$$\Rightarrow |a_n|^{1/n} > \frac{1}{\rho} \left[\text{as } 1/\rho < 1/R, \text{ for every } \epsilon > 0, \text{ we take } 1/\rho = 1/R - \epsilon\right]$$

$\Rightarrow |a_n z^n| > \left(\frac{|z|}{\rho}\right)^n$ for infinitely many n.

$\Rightarrow$ terms of the series $\sum |a_n z^n|$ are unbounded as $|z| > R$

$\Rightarrow$ the series $\sum a_n z^n$ is divergent.

(iii) For $0 \leq \rho < R$, we choose a (+) real no. ρ' such that $\rho < \rho' < R$.

So, $\frac{1}{R} < \frac{1}{\rho'} < \frac{1}{\rho}$.

From definition of $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1/R$

$$\Rightarrow |a_n|^{1/n} \leq 1/R + \epsilon \ \forall n \geq m$$

$$\Rightarrow |a_n|^{1/n} \leq 1/\rho' \left[\text{as } 1/\rho' > 1/R, \text{ for every } \epsilon > 0, \text{ we can take } 1/\rho' = 1/R + \epsilon\right]$$

$$\Rightarrow |a_n| \leq 1/\rho'^n \Rightarrow |a_n z^n| \leq \frac{|z^n|}{\rho'^n} \leq \left(\frac{\rho}{\rho'}\right)^n$$

for $|z| \leq \rho$

~~Let~~

Let $M_n = \left(\frac{\rho}{\rho'}\right)^n$

Then $\sum M_n = \sum \left(\frac{\rho}{\rho'}\right)^n$ is convergent being a G.P. series with common ratio $\frac{\rho}{\rho'} < 1$.

Therefore, by ~~Wiei Weiel~~ Weierstrass M-Test $\sum a_n z^n$ converges uniformly for $|z| \leq \rho' < R$.

* Defn Weierstrass's test for uniform convergence of a P.S. $\sum a_n z^n$: (Weierstrass M-test).

A power series $\sum a_n z^n$ converges uniformly ~~on~~ $\forall z \in A$, if (i) $|a_n z^n| \leq M_n \ \forall n \in N$, where M_n is a (+)ve constant (independent of z) (ii) $\sum M_n$ is convergent.

* Theorem :- The power series $\sum_1^\infty n a_n z^{n-1}$, obtained by differentiating term by term the power series $\sum_0^\infty a_n z^n$ has the same ~~sid~~ radius of convergence as that of $\sum a_n z^n$.

Proof :- Let, R, R' be the radius of convergence of the power series $\sum_0^\infty a_n z^n$ and $\sum_1^\infty a_n z^{n-1}$ respectively.

Then $\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$ and $\frac{1}{R'} = \overline{\lim_{n\to\infty}} |n a_n|^{1/n}$.

Therefore, $\frac{1}{R'} = \overline{\lim_{n\to\infty}} n^{1/n} \cdot |a_n|^{1/n} = \frac{1}{R} \cdot \lim_{n\to\infty} n^{1/n}$ ——— (1)

Now, by Cauchy's limit theorem,

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ and so $\lim_{n\to\infty} n^{1/n}$

$= \lim_{n\to\infty} \frac{n+1}{n}$

$= 1.$

Hence, from (1),

$\cancel{1/R' = 1/R}$ $1/R' = 1/R$, i.e., $R = R'$.

For integration

* Theorem:-

The power series $\sum_{1}^{\infty} a_n \frac{z^{n+1}}{n+1}$ obtained by integrating term by term the power series $\sum_{0}^{\infty} a_n z^n$ has the same radius of convergence as that of $\sum a_n z^n$.