C-10 unit-4

Vector spaces:

A vector spaces or linear spaces consists..

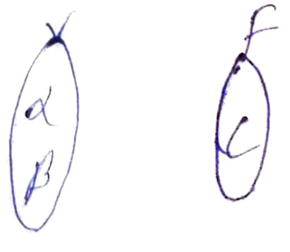
1) A field of scalar (F)
2) A set V of objects of vector
3) A operation called vector addition for which V is additive abelian group.
4) A operation called scalar multiplication which satisfies

i) $1\alpha = \alpha$ for every α in V

ii) $(c_1 c_2)\alpha = c_1(c_2\alpha)$, $c_1, c_2 \in F$, $\alpha \in V$

iii) $c(\alpha+\beta) = c\alpha + c\beta$, $c \in F$, $\alpha, \beta \in V$

iv) $(c_1+c_2)\alpha = c_1\alpha + c_2\alpha$, $c_1, c_2 \in F$, $\alpha \in V$



Linearly independent

Let, V be a vector space over F. A subset $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is called linearly independent if there exist some scalar $c_1, c_2, \ldots, c_n$. such that ~~$c_1+\alpha_1$~~ $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0$, $c_i \neq 0$ for some i, $\alpha_i \neq 0 \; \forall i$.

A set which is not linearly dependent is called linearly independent.

~~Linear tra~~

Basis

Let, $V(F)$ be a vector spaces a ~~finite~~ set S of V ~~of V~~ ~~$S\{V\}$~~ is said to be basis of V if

i) S is linearly independent

ii) S generates V

Dimension of a vector spaces:

dimension of $V(F)$ or $\dim(V(F))$ = Number of elements in basis set of V

Linear transformation

Let, 'V' and 'W' be vector spaces over the field F. A linear transformation from 'V' into 'W' 'U' is a function T : $V \xrightarrow{\text{into}} U$ by

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta), \ \forall\, a, b \in F, \alpha, \beta \in V$$

or

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta); \ \forall\, c \in F, \alpha, \beta \in V$$

Q) Which of the following function T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations!

a) $T(x_1, x_2) = (1 + x_1, x_2)$;

Soln: Given,

~~$T(x_1, x_2$~~ $T : \mathbb{R}^2$ into $\mathbb{R}^2$ by the rule

$$T(x_1, x_2) = (1 + x_1, x_2)$$

Now, let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2) \in \mathbb{R}$

and $c \in \mathbb{R}$ be a scalar

Now, $T(c\alpha + \beta) = T(c(x_1, x_2) + (y_1, y_2))$

$= T\{c(x_1, x_2) + (y_1, y_2)$

$= T((cx_1, cx_2) + (y_1, y_2))$

~~$= T(1 + cx_1 +$~~

$= T(cx_1 + y_1, cx_2 + y_2)$

$= (1 + cx_1 + y_1, cx_2 + y_2)$

$= (1 + cx_1, cx_2) + (y_1, y_2)$

$\neq cT(\alpha) + T(\beta)$

So, T is not a linear transformation

b) $T(x_1, x_2) = (x_2, x_1)$

Sol^n: let, $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2) \in \mathbb{R}^2$

$c \in \mathbb{R}$

$T(c\alpha + \beta) = T(c(x_1, x_2) + (y_1, y_2))$

$= T(cx_1 + y_1, cx_2 + y_2)$

$= (cx_2 + y_2, cx_1 + y_1)$

$= (cx_2, cx_1) + (y_2, y_1)$

$= c(x_2, x_1) + (y_2, y_1)$

$= cT(x_1, x_2) + T(y_1, y_2)$

$= cT(\alpha) + T(\beta)$

Hence T is a linear transformation

c) $T(x_1, x_2) = (x_1 - x_2, 0)$.

Sol^n: Given, $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by the scale

$T(x_1, x_2) = (x_1 - x_2, 0)$

Let, $\alpha = (x_1, x_2), \beta = (y_1, y_2) \in \mathbb{R}^2$

$c \in \mathbb{R}$ be a scalar

Now, $T(c\alpha + \beta) = T(c(x_1, x_2) + (y_1, y_2))$

$= T(cx_1 + y_1, cx_2 + y_2)$

$= (cx_1 + y_1 - (cx_2 - y_2, 0)$

$= (\cancel{cx_1, cx_2) - (y_1, y_2)}$

$\cancel{c(x_1, x_2) - (y_1, y_2)}$

$= \big(cx_1 - cx_2 + y_1 - y_2 \, , \, 0 + 0 \big)$

$= \big(c(x_1 - x_2) + (y_1 - y_2) \, , \, 0 + 0 \big)$

$= \big(c(x_1 - x_2), c0 \big) + \big(y_1 - y_2 , 0 \big)$

$= c\big(x_1 - x_2 , 0 \big) + \big(y_1 - y_2 , 0 \big)$

$= c\,T(x_1, x_2) + T(y_1, y_2)$

$= c\,T(\alpha) + T(\beta)$ So T is a L.T.

Ans

Q) The mape $T: V \to W$ defined by $T(v) = 0$ for all $v \in V$ is a linear transformation.

Soln: Given, $T: V \to W$ by

$$T(v) = 0 \ \forall \ v \in V$$

Let, $u, v \in V$, c be any scalar

$$\therefore T(cu+v) = 0 \quad (\because T(v) = 0)$$
$$= c.0 + 0$$
$$= cT(u) + T(v)$$

$\therefore T$ is a linear transformation.

Q) Let, $T: R^3 \to R^3$ defined by $T(x,y,z) = (x,y,0)$ $\forall (x,y,z) \in \mathbb{R}^3$ prove that T is L.T.

Soln: Given, $T: R^3 \to \mathbb{R}^3$ by

$$T(x,y,z) = (x,y,0) \ \forall (x,y,z) \in \mathbb{R}^3$$

Q) Let, $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by i) $T(x,y,z) = (x,y,0)$
$\forall (x,y,z) \in \mathbb{R}$. prove that T is L.T.

Sol: Given,

$T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$T(x,y,z) = (x,y,0) \; \forall (x,y,z) \in \mathbb{R}^3$

Let, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$c \in \mathbb{R}$ be a scalar

$\cancel{T(c\alpha+\beta) = cT(\alpha}$

$$T(c\alpha+\beta) = T(c(\alpha)+\beta)$$

$$= T(c(x_1, y_1, z_1) + (x_2, y_2, z_2))$$

$$= T(cx_1 + x_2, cy_1 + y_2, cz_1 + z_2)$$

$$= (cx_1 + x_2, cy_1 + y_2, 0)$$

$$= (cx_1 + x_2, cy_1 + y_2, c.0 + 0)$$

$$= (cx_1, cy_1, c.0) + (x_2, y_2, 0)$$

$$= c(x_1, y_1, 0) + (x_2, y_2, 0)$$

$$= T(c(x_1, y_1, z_1)) + T(x_2, y_2, z_2)$$

$$= T(c(\alpha)) + T(\beta)$$

$\therefore$ T is L.T.

ii) $T(x,y,z) = (x+y+z, 2x-3y+4z) \; \forall (x,y,z) \in \mathbb{R}^3$

Soln: Given, $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$T(x,y,z) = (x+y+z, 2x-3y+4z) \; \forall (x,y,z) \in \mathbb{R}^3$

Let, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$, $c \in \mathbb{R}$ be a scalar

$T(c\alpha+\beta) = T((c\alpha)+\beta)$

$= T(c(x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= T(cx_1+x_2, cy_1+y_2, cz_1+z_2)$

$= (cx_1+x_2+cy_1+y_2+cz_1+z_2, 2(cx_1+x_2)-3(cy_1+y_2)+4(cz_1+z_2))$

$= (c(x_1+y_1+z_1)+x_2+y_2+z_2, 2cx_1-3cy_1+4cz_1+2x_2-3y_2+4z_2)$

$= (c(x_1+y_1+z_1), c(2x_1-3y_1+4z_1)) + (x_2+y_2+z_2, 2x_2-3y_2+4z_2)$

$= c(x_1+y_1+z_1, 2x_1-3y_1+4z_1) + (x_2+y_2+z_2, 2x_2-3y_2+4z_2)$

$= cT(x_1+y_1+z_1) + T(x_2+y_2+z_2)$

$= cT(\alpha) + T(\beta)$

$\therefore T$ is linear transformation

Subspace

Let, $V(F)$ be a vector space over the field F and $W \subseteq V$ then W is called the subspace of $V(F)$ over same field F if $\alpha, \beta \in V$, $a, b \in F$

$\Rightarrow a\alpha + b\beta \in V$.

Kernal and image of a linear transformation

Let, $T: V \to U$ be a linear transformation. The Kernal of T writen $Ker T$ is the set of elements in V that map into the zero vector 0 in U. i.e.

$~~Ket~~$ $Ker T = \{v \in V : T(v) = 0$

~~Kernal is also known as image or range~~

and ~~Ker T is~~

Image or range

$T: V \to U$ be a L.T. the image or range of T writen $im(T)$ is the set of elements in U i.e.

$Im T = \{u \in U : \exists v$ v for which $T(v) = u\}$.

Range and Nullspace of a linear transformation.

Let, $T: U(F) \to V(F)$ be a linear transformation then the range of T. ~~R(T) or Range(T)~~

$R(T)$ or $Range(T) = \{T(\alpha) \in V : \alpha \in U\}$

Null space

Let, $T: U(F) \to V(F)$ be a linear transformation then ~~the~~ nullspace of T.

~~N(T) or Null(T) =~~

$N(T) = \{\alpha \in U : T(\alpha) = 0 \in V\}$

Theorem: If $U(F)$ and $V(F)$ are two vector spaces and T is a linear transformation ~~from U into V~~ of U into V then range of T is a subspace of V.

Proof: Let, $\beta_1, \beta_2 \in R(T)$

Then, $\exists\, \alpha_1, \alpha_2 \in U : T(\alpha_1) = \beta_1,\ T(\alpha_2) = \beta_2$

Let, a, b be any two scalars

Now, $\cancel{a(\beta_1) + b(\beta_2) = a(T(\alpha_1)) + b\{T(\alpha_2)\}}$

Now, $a\beta_1 + b\beta_2 = aT(\alpha_1) + bT(\alpha_2)$

$= T(a\alpha_1 + b\alpha_2)$ [$\because$ T is L.T]

Now, U is a vector space

$a\alpha_1 + b\alpha_2 \in U \quad \forall \alpha_1, \alpha_2 \in U$

Consequently,

$$T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2 \in R(T)$$

So $R(T)$ is a subspace of $V(F)$.

Theorem: If $U(F)$ and $V(F)$ are two vector spaces and T is a linear transformation from U into V then ~~the~~ Nullspace of T or Kernel of T is a subspace of T.

Proof: Given, $T : U(F) \rightarrow V(F)$ and,

$$N(T) = \{\alpha \in U : T(\alpha) = 0 \in V\}$$

$\because T(0) = 0 \in V$

$\therefore 0 \in N(T) \neq \phi$

Let, $\alpha_1, \alpha_2 \in N(T)$

$\Rightarrow T(\alpha_1) = 0, T(\alpha_2) = 0$

Let, $a, b \in F$

Now, $a\alpha_1 + b\alpha_2 \in U$ and $T(a\alpha_1 + b\alpha_2)$

$$= aT(\alpha_1) + bT(\alpha_2)$$

$$= a.0 + b.0$$

$$= 0 \in V$$

$\therefore N(T)$ or $Ker(T)$ is a subspace of T

Hence proved

Rank and nullity of a linear transformation. Definition

Let T be a linear transformation ~~from~~ from a vector space $U(F)$ into a vector space $V(F)$ with U as finite dimensional. The rank of T denoted by $\rho(T)$ is the dimension of the range of T i.e.

$$\rho(T) = \dim R(T).$$

The nullity of T denoted by $\nu(T)$ is the dimension of the null space of T i.e. $\nu(T) = \dim N(T)$.

Note: some standard basis

1) For $\mathbb{R}^2$ or $V_2(\mathbb{R})$ ~~sero~~ basis is

$\{(1,0),(0,1)\}$

ii) For $\mathbb{R}^3$ or $V_3(\mathbb{R})$ basis is

$\{(1,0,0),(0,1,0)(0,0,1)\}$

Q) Show that the mapping $T: V_2(R) \to V_3(R)$ defined as
$T(a,b) = (a+b, a-b, b)$
is a linear transformation from $V_2(R)$ into $V_3(R)$.
Find the range, rank, nullspace and nullity of T.

Sol^n Given, $T: V_2(R) \to V_3(R)$ by

$T(a,b) = (a+b, a-b, b)$

Let $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2) \in V_2(R)$

~~a, b ∈ R be a scalar~~

$a, b \in R$ be any two scalar

$T(a\alpha + b\beta) = T(a(\alpha) + b(\beta))$

$= T(a(a_1, b_1) + b(a_2, b_2))$

$= T(aa_1, ab_1 + ba_2, bb_2)$

$\cancel{= T(aa_1 + ba_2 ; ab_1 + bb_2)}$

$\cancel{= (aa_1, ab_1 + ba_2, bb_2, aa_1, ab_1 - ba_2, bb_2, ba_2 bb_2)}$

$= T(aa_1 + ba_2, ab_1 + bb_2)$

$= (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, ab_1 + bb_2)$

$= (aa_1 + ab_1, aa_1 - ab_1, ab_1) + (ba_2 + bb_2, ba_2 - bb_2, bb_2)$

$= a(a_1 + b_1, a_1 - b_1, b_1) + b(a_2 + b_2, a_2 - b_2, b_2)$

$= aT(a_1, b_1) + bT(a_2, b_2)$

$= aT(\alpha) + bT(\beta)$

$\therefore T$ is a L.T.

We know that,

Basis of $V_2(R)$ is $\{(1,0),(0,1)\}$

Now, $T(1,0) = (1+0, 1-0, 0) = (1,1,0)$

$T(0,1) = (0+1, 0-1, 1) = (1,-1,1)$.

$T(1,0) = (1,1,0), T(0,1) = (1,-1,1)$ generates range of T

Now, we have to show that.

$(1,1,0), (1,-1,1)$ are linearly independent in range(T).

Let, $c_1, c_2 \in R$

$c_1(1,1,0) + c_2(1,-1,1) = 0$

$\Rightarrow (c_1, c_1, 0) + (c_2, -c_2, c_2) = 0$

$\Rightarrow (c_1 + c_2, c_1 - c_2, c_2) = (0, 0, 0)$

$\Rightarrow c_1 + c_2 = 0, \; c_1 - c_2 = 0, \; c_2 = 0$

$c_1 + 0 = 0$

$\Rightarrow c_1 = 0$

$\therefore c_1 = 0$ & $c_2 = 0$

$\therefore (1,1,0), (1,-1,1)$ are linearly independent

$\therefore$ Basis of range $(T) = \{(1,1,0), (1,-1,1)\}$

$\dim R(T) = \rho(T) = 2$

Again, to find nullspace $N(T)$,

Let, $(a,b) \in N(T)$

$\Rightarrow T(a,b) = 0$

$\Rightarrow (a+b, a-b, b) = (0,0,0)$

$\Rightarrow a+b = 0, \; a-b = 0, \; b = 0$

$\Rightarrow a = 0$

$\therefore a = 0, b = 0$

So, $(0,0) \in N(T)$ is only element of $N(T)$

$\therefore N(T) = \{(0,0)\}$

$\dim(N(T)) = 0 = \text{Null}(T)$

Q) Let F be field of complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2).$$

Verify that T is a linear transformation. Describe the null space of T.

Soln: Given,

$$T : F^3 \to F^3 \text{ by}$$

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2)$$

Let, $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in \mathbb{R}^3$

$c \in \mathbb{R}$ be a scalar

$$\therefore T(c\alpha + \beta) = T(c(\alpha) + \beta)$$

$$= T(c(x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$= T((cx_1, cx_2, cx_3) + (y_1, y_2, y_3))$$

$$= T(cx_1 + y_1, cx_2 + y_2, cx_3 + y_3)$$

$$= (cx_1 + y_1 - cx_2 - y_2 + 2cx_3 + 2y_3, 2cx_1 + 2y_1 + cx_2 + y_2 - cx_3 - y_3, -cx_1 - y_1 - 2cx_2 - 2y_2)$$

$$= \big(cx_1 - cx_2 + 2cx_3 + y_1 - y_2 + 2y_3,\ 2cx_1 + cx_2 - cx_3 + 2y_1 + y_2 - y_3,$$
$$-cx_1 - 2cx_2 - y_1 - 2y_2\big)$$

$$= \big(c(x_1 - x_2 + 2x_3) + y_1 - y_2 + 2y_3,\ c(2x_1 + x_2 - x_3) + 2y_1 + y_2 - y_3,$$
$$c(-x_1 - 2x_2) - y_1 - 2y_2\big)$$

$$= \big(c(x_1 - x_2 + 2x_3), c(2x_1 + x_2 - x_3), c(-x_1 - 2x_2)\big) + \big(y_1 - y_2 + 2y_3, 2y_1$$
$$+ y_2 - y_3, -y_1 - 2y_2\big)$$

$$= c\big(x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2\big) + \big(y_1 - y_2 + 2y_3, 2y_1 + y_2$$
$$- y_3, -y_1 - 2y_2\big)$$

$$= cT(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$

$$= cT(\alpha) + T(\beta)$$

$\therefore$ T is a linear transformation

Now, $(x_1, x_2, x_3) \in N(T)$

$$\Leftrightarrow T(x_1, x_2, x_3) = (0, 0, 0)$$

$$\Leftrightarrow (x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_3) = (0, 0, 0)$$

$$\Leftrightarrow \left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \\ -x_1 - 2x_2 + 0x_3 &= 0. \end{aligned}\right\} \longrightarrow (1)$$

$\therefore$ The null space of T is the soln space of the system of ~~L.T~~ linear homogeneous eqn (2)

Let, A be the coefficient matrix of eqn (1)

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\underset{\sim}{\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & -3 & 2 \end{bmatrix}$$

$$\underset{\sim}{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & -3 \end{bmatrix}$$

$\therefore$ Rank(A) = 3 = no.s of unknown

Hence, the only solution of system (1) is $x_1 = x_2 = x_3 = 0$

$\therefore$ $N(T) = \{(0,0,0)\}$

Null$(T) = 0$

Q) Show that the mapping $T: R^2 \to R^3$ defined as
$T(a,b) = (a-b, b-a, -a)$
is a L.T from R^2 into R^3. Find the range, rank, null-space and nullity of T.

<u>Soln</u>: Given,
$T: R^2 \to R^3$ by
$T(a,b) = (a-b, b-a, -a)$

let, $\alpha = (a_1, b_1)$ $\beta = (a_2, b_2) \in R^2$
$c \in R$ be a scalar

$$T(c\alpha+\beta) = T(c(\alpha)+\beta)$$
$$= T(c(a_1,b_1)+(a_2,b_2))$$
$$= T((ca_1, cb_1)+(a_2,b_2))$$
$$= T(ca_1+a_2, cb_1+b_2)$$
$$= (ca_1+a_2-cb_1-b_2, cb_1+b_2-ca_1-a_2, -ca_1-a_2)$$
$$= (c(a_1-b_1)+a_2-b_2, c(b_1-a_1)+b_2-a_2, -ca_1-a_2)$$
$$= c(a_1-b_1, b_1-a_1, -a_1) + (a_2-b_2, b_2-a_2, -a_2)$$
$$= cT(a_1,b_1) + T(a_2,b_2)$$
$$= cT(\alpha) + T(\beta)$$

$\therefore T$ is a L.T.

We know that,

Basis of R^2 is $\{(1,0),(0,1)\}$

Now, $T(1,0) = (1-0, 0-1, -1) = (1,-1,-1)$

$T(0,1) = (0-1, 1-0, -0) = (-1,1,0)$

$T(1,0) = (1,-1,-1)$, $T(0,1) = (-1,1,0)$ generates range of T

Now, we have to show that,

$(1,-1,-1), (-1,1,0)$ are linearly independent in range(T).

Let, $c_1, c_2 \in R$

$c_1(1,-1,-1) + c_2(-1,1,0) = (0,0,0)$

$\Rightarrow (c_1, -c_1, -c_1) + (-c_2, c_2, 0) = (0,0,0)$

$\Rightarrow (c_1 - c_2, -c_1 + c_2, -c_1) = (0,0,0)$

$c_1 - c_2 = 0$, $-c_1 + c_2 = 0$, $-c_1 = 0 \Rightarrow c_1 = 0$

$0 - c_2 = 0$

$c_2 = 0$

$\therefore c_1 = 0$ & $c_2 = 0$

$\therefore (1,-1,-1), (-1,1,0)$ are linearly independent

$\therefore$ Basis of Range $(T) = \{(1,-1,-1), (1,-1,1)\}$

~~Rank~~ $\dim R(T) = \rho(T) = 2$

Again, To find Nullspace $N(T)$

Let, $(a,b) \in N(T)$

$\Rightarrow T(a,b) = 0$

$\Rightarrow (a-b, b-a, -a) = (0,0,0)$

$a-b=0, \; b-a=0, \; -a=0$

$a=0$

$\Rightarrow 0-b=0$

$\Rightarrow b=0$

$\therefore a=0, \; b=0$

So, $(0,0) \in N(T)$ is only element of $N(T)$

$\therefore N(T) = \{(0,0)\}$

$\dim N(T) = 0 = \text{Nullity}(T)$

Matrix representation of a linear transformation

Let, $U(F)$ and $V(F)$ be two n-dimensional and m-dimensional vector spaces.

Let,

$$B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

$$B' = \{\beta_1, \beta_2, \ldots, \beta_m\}$$

be ordered basis for U and V respectively.

Let, $T : U \to V$ be a linear transformation

each of 'n' vector $T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + \ldots + a_{mj}\beta_m$

The scalars $a_{1j}, a_{2j}, \ldots, a_{mj}$ are the co-ordinates of $T(\alpha_j)$ in the ordered basis B'. The $m \times n$ matrix whose j^{th} column $(j = 1, 2, \ldots, n)$ consists of these co-ordinate is called the matrix of the linear transformation T relative to the pair of ordered bases B and B'.

We shall denote it by the symbol

$[T; B; B']$ or simply by $[T]$ if the bases are understood.

NB1:

$$T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \dots + a_{m1}\beta_m$$

$$T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \dots + a_{m2}\beta_m$$

$$T(\alpha_3) = a_{13}\beta_1 + a_{23}\beta_2 + \dots + a_{m3}\beta_m$$

$$\vdots$$

$$T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \dots + a_{mn}\beta_m$$

$$[T; B; B'] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & a_{1n} \\ a_{21} & a_{22} & & a_{2j} & a_{2n} \\ a_{31} & a_{32} & & a_{3j} & a_{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & a_{mn} \end{bmatrix}_{m \times n}$$

(P-159)

(10) Let T be a L.T on the vector space $V_2(F)$ defined by $T(a,b) = (a,0)$.

Write the matrix of relative to the standard ordered basis of $V_2(F)$.

Sol^n^: Let, $B = \{\alpha_1 = (1,0), \alpha_2 (0,1)\}$ be the standard order basis for $V_2(F)$

It is given that $T: V_2(F) \longrightarrow V_2(F)$

$$T(a,b) = (a,0)$$

$$T(\alpha_1) = T(1,0) = (1,0) = 1(1,0) + 0(0,1)$$

$$T(\alpha_2) = T(0,1) = (0,0) = 0(1,0) + 0(0,1)$$

So the matrix representation of T is

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Suppose T is a L.T from $\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-1\end{bmatrix}$, and $T\left\{\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right\} = \begin{bmatrix}0\\1\end{bmatrix}$ find the formula for the image of an arbitrary α in $\mathbb{R}^2$

Sol^n: Given,

$$T: \mathbb{R}^2 \to \mathbb{R}^2 \text{ and,}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

Let, $\alpha \in \mathbb{R}^2$

$$\alpha = (x_1, x_2)$$

$$= \{x_1(1,0) + x_2(0,1)\}$$

$$T(\alpha) = T\left(x_1(1,0) + x_2(0,1)\right)$$

$$= x_1\{T(1,0)\} + x_2\{T(0,01)\}$$

$$= x_1(2,-1) + x_2(0,1)$$

$$= (2x_1, -x_1) + (0, x_2)$$

$$T(x_1, x_2) = (2x_1, -x_1 + x_2)$$

Property of Linear transformation

Let, T be a linear transformation from U(F) into V(F) then show that

i) $T(0) = 0$

ii) $T(-\alpha) = T(\alpha)$

iii) $T(\alpha - \beta) = T(\alpha) - T(\beta)$

Proof: i) Let, $\alpha \in U$

$\Rightarrow T(\alpha) \in V.$

Now, $T(\alpha) + 0 = T(\alpha)$

$\Rightarrow T(\alpha) + 0 = T(\alpha + 0)$

$\Rightarrow T(\alpha) + 0 = T(\alpha) + T(0)$ [$\because$ T is a L.T]

$\Rightarrow 0 = T(0)$ [by left cancellation]

So, $T(0) = 0$

ii) Since, $T(\alpha + (-\alpha)) = T(\alpha) + T(-\alpha)$

$\Rightarrow T(0) = T(\alpha) + T(-\alpha)$

$\Rightarrow 0 = T(\alpha) + T(-\alpha)$

$\Rightarrow T(-\alpha) = -T(\alpha)$

$a + b = 0$
$\Rightarrow a = -b$
$\Rightarrow -a = b$

iii) $T(\alpha - \beta) = T(\alpha + (-\beta))$

$\Rightarrow T(\alpha - \beta) = T(\alpha) + T(-\beta)$ [$\because$ T is L.T]

$\Rightarrow T(\alpha - \beta) = T(\alpha) - T(\beta)$ [according to property ii]

Hence proved

Algebra of linear transformation:

If U, V and W be the vector space over the same field F. Let T is linear transformation from U into V and X be the linear transformation from V into W. Then, XT (Product of linear transformation) defined by

$$(XT)(\alpha) = X[T(\alpha)] \ \forall \alpha \in U$$

is ~~A~~ L.T from U into W.

Proof: Since, $T : U \to V$ and $X : V \to W$.

Then, $\alpha \in U \Rightarrow T(\alpha) \in V$

and also $T(\alpha) \in V \Rightarrow X[T(\alpha)] \in W$

i.e. $(XT)(\alpha) \in W$

It means XT is function from U into W. Now, We shall prove that XT is a L.T. from U into W.

For this, let, $a, b \in F$ and $\alpha, \beta \in U$

$$\therefore XT(a\alpha + b\beta) = X[T(a\alpha + b\beta)]$$

$$= X[aT(\alpha) + bT(\beta)] \quad [\because T \text{ is L.T}]$$

$$\cancel{aX[T(\alpha)] = bX[T(\beta)]} \quad \cancel{[\because T \text{ and } X \text{ is L.T.}]}$$

$$= a(XT)(\alpha) + b(XT)(\beta) \quad [\because X \text{ is L.T}]$$

Hence, XT is linear transformation from U into W

Hence proved

Linear operator

A mapping $T: V(F) \to V(F)$ by T:

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$, $\forall a, b \in F$, $\alpha, \beta \in V$

is called a linear operator (L.O).

Linear Transformation

A mapping $T: U(F) \to V(F)$ by

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$, $\forall a, b \in F$, $\alpha, \beta \in V$

is called a linear transformation (L.T.).

Q) Let, T_1 and T_2 be two linear operators on $\mathbb{R}^2$ defined as $T_1(a_1, a_2) = (a_2, a_1)$ and $T_2(a_1, a_2) = (a_1, 0)$ then show that $T_1 T_2 \neq T_2 T_1$

Soln: Given, T_1 and T_2 are linear operator on $\mathbb{R}^2$ defined by

$$T_1(a_1, a_2) = (a_2, a_1)$$
$$T_2(a_1, a_2) = (a_1, 0)$$

$$\begin{aligned} T_1 T_2 (a_1, a_2) &= T_1 [T_2(a_1, a_2)] \\ &= T_1(a_1, 0) \\ &= (0, a_1) \end{aligned}$$

$$\begin{aligned} T_2 T_1 (a_1, a_2) &= T_2 [T_1(a_1, a_2)] \\ &= T_2 (a_2, a_1) \\ &= (a_2, 0) \end{aligned}$$

$\therefore T_1 T_2 \neq T_2 T_1$ Hence proved

Q) Let, $V(\mathbb{R})$ be the vector space of all polynomial in x with co-efficient in the field $\mathbb{R}$. Let, D ~~be~~ and T be two L.T. on V defined as

$D[f(x)] = \frac{d}{dx} f(x)$, $\forall f(x) \in V$ and

$T[f(x)] = xf(x)$, $\forall f(x) \in V$ then show that

$DT \neq TD$

Soln: Given, $V(\mathbb{R})$ is vector space

D and T are two L.T on V defined by

$D[f(x)] = \frac{d}{dx} f(x)$, $\forall f(x) \in V$

$T[f(x)] = xf(x)$, $\forall f(x) \in V$

We need to show that

$DT \neq TD$

$$\therefore DT[f(x)] = D[T(f(x))]$$
$$= D(xf(x))$$
$$= \frac{d}{dx} xf(x) = x\frac{d}{dx} f(x) + f(x) \frac{d}{dx} x$$
$$= xf'(x) + f(x)$$

Again, $TD[f(x)] = T[D(f(x))]$
$$= T\left[\frac{d}{dx} f(x)\right]$$
$$= T[f'(x)]$$
$$= xf'(x)$$

$\therefore DT \neq TD$ Hence proved

Q Let T be a linear operator on ~~R(I)~~ $\mathbb{R}^2$ defined by $T(x,y) = (4x-2y, 2x+y)$ compute the matrix of T relative to the basis (α_1, α_2) where $\alpha_1 = (1,1)$, $\alpha_2 = (-1,0)$

Solⁿ :- Given,

$$T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R}) \text{ by}$$

$$T(x,y) = (4x-2y, 2x+y)$$

$$T(\alpha_1) = T(1,1)$$
$$= (4\cdot1 - 2\cdot1,\ 2\cdot1+1)$$
$$= (2,3)$$

$$T(\alpha_2) = T(-1,0)$$
$$= (4(-1)-0,\ 2(-1)+0)$$
$$= (-4,-2)$$

$$\cancel{\text{Now, } T(\alpha_1) = (2,3) = a(1,0) + b(0,1)}$$
$$\cancel{= 2(1,0) + 3(0,1)}$$

$$\cancel{\text{Sly, } T(\alpha_2) = (-4,-2) = a(1,0) + b(0,1)}$$
$$\cancel{= -4(1,0) + (-2)(0,1)}$$

Now, $T(\alpha_1) = (2,3) = a(1,1) + b(-1,0)$

$$\Rightarrow (2,3) = (a-b, a)$$
$$\Rightarrow a = 3, \Rightarrow a-b = 2 \Rightarrow b = 1$$

$$\therefore \cancel{T(\alpha_1) = 3,1}$$

$\therefore T(\alpha_1) = (2,3) = 3(1,1) + 1(-1,0)$ — (i)

Slly, $T(\alpha_2) = (-4,-2) = a(1,1) + b(-1,0)$

$\Rightarrow (-4,-2) = (a-b, a)$

$\Rightarrow a = -2,\ a-b = -4 \Rightarrow b = 2$

$\therefore T(\alpha_2) = (-4,-2) = -2(1,1) + 2(-1,0)$ — (ii)

$$\therefore [T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

Q) Let T be a linear operation on $\mathbb{R}^2$ defined by $T(x,y) = (2y, 3x-y)$ Find the matrix representation of T relative to the basis on $\{(1,3),(2,5)\}$

Sol^n: Given,

$T: \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$

$T(x,y) = (2y, 3x-y)$

$T(\alpha_1) = T(1,3)$

$= (6,0)$

$T(\alpha_2) = T(2,5)$

$= (10,1)$

Now, $T(\alpha_1) = (6,0) = a(1,3) + b(2,5)$

$\Rightarrow (6,0) = (a+2b, 3a+5b)$

$\Rightarrow a+2b = 6 \; ; \; 3a+5b = 0$

$\Rightarrow a = 6-2b \; ; \; 3(6-2b)+5b = 0$

$\Rightarrow a = \cancel{6}-30 \qquad 18-6b+5b = 0$

$\Rightarrow b = 18$

$\therefore T(\alpha_1) = (6,0) = -30(1,3) + 18(2,5)$

Slly $T(\alpha_2) = (10,1) = a(1,3) + b(2,5)$

$\Rightarrow (10,1) = (a+2b, 3a+5b)$

$\Rightarrow a+2b = 10 \; ; \; 3a+5b = 1$

$\Rightarrow a = 10-2b \; ; \; 3(10-2b)+5b = 1$

$\Rightarrow a = 10-58 \qquad \Rightarrow 30-6b+5b = 1$

$\Rightarrow a = -48 \qquad \Rightarrow b = 29$

$\therefore T(\alpha_2) = (10,1) = -48(1,3) + 29(-1,0)$

$$\therefore [T]_B = \begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$$

Q5) Let T be a linear operator $V_3(\mathbb{R})$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c) \; \forall (a,b,c) \in V_3(\mathbb{R})$. Is T invertible? If so find rule for T^{-1} like the one which defines T.

Soln: Given,

$$T : V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R}) \text{ by}$$

$$T(a,b,c) = (3a, a-b, 2a+b+c) \; \forall (a,b,c) \in V_3(\mathbb{R})$$

One-one

Let, $\alpha = (a_1, b_1, c_1)$ and $\beta = (a_2, b_2, c_2) \in V_3(\mathbb{R})$

Then, $T(\alpha) = T(\beta)$

$\Rightarrow T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$

$\Rightarrow (3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$

$\Rightarrow 3a_1 = 3a_2 \; ; \; a_1 - b_1 = a_2 - b_2, \; 2a_1 + b_1 + c_1 = 2a_2 + b_2 + c_2$

$\Rightarrow a_1 = a_2 \; ; \; \Rightarrow b_1 = b_2 \; ; \; \Rightarrow c_1 = c_2$

$\Rightarrow (a_1, b_1, c_1) = (a_2, b_2, c_2)$

$\Rightarrow \alpha = \beta$

So, T is one-one

Onto

T is a linear transformation on $V_3(\mathbb{R})$ and

$\dim(V_3(\mathbb{R})) = 3$.

$\because$ T is one-one

So, T is onto

Hence T is invertible

If, $T(a,b,c) = (p,q,r)$

$\Rightarrow (a,b,c) = T^{-1}(p,q,r)$ — (1)

Now, $T(a,b,c) = (p,q,r)$

$\Rightarrow (3a, a-b, 2a+b+c) = (p,q,r)$

$\Rightarrow 3a = p \; ; a-b = q , \; 2a+b+c = r$

$\Rightarrow a = \frac{p}{3} \; ; \Rightarrow -b = q - \frac{p}{3} , \; 2\frac{p}{3} \cancel{- \frac{3q-p}{3}} + c = r$

$\cancel{\Rightarrow b = \left(\frac{3q-p}{3}\right)} \qquad \Rightarrow c = r - \frac{2p}{3} - \frac{3q-p}{3}$

$\Rightarrow b = \frac{p}{3} - q \qquad = r -$

$\Rightarrow 2\frac{p}{3} + \frac{p}{3} - q + c = r$

$\Rightarrow c = r - p + q$

$\therefore$ (1) $\Rightarrow T^{-1}(p,q,r) = \left(\frac{p}{3}, \frac{p}{3} - q, r - p + q\right)$ which is the required rule of T^{-1}

Q) Let, F be any field. Let T be a linear operator on F^2 defined by $T(a,b) = (a+b, a)$ show that T is invertible and find the rule for T^{-1}

Sol^n: Given,

$$T : F^2 \to F^2 \text{ by}$$

$$T(a,b) = (a+b, a)$$

one-one,

Let, $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2) \in F^2$

then, $T(\alpha) = T(\beta)$

$\Rightarrow T(a_1, b_1) = T(a_2, b_2)$

$\Rightarrow (a_1+b_1, a_1) = (a_2+b_2, a_2)$

$\Rightarrow a_1+b_1 = a_2+b_2$; $a_1 = a_2$

$\Rightarrow b_1 = b_2$; $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

$\Rightarrow \alpha = \beta$

So, T is one-one

Onto

T is a linear transformation on F^2

$\dim(F^2) = 2$

$\because T$ is one-one

So, T is onto

Hence T is invertible

If $T(a,b) = (p,q)$

$\Rightarrow (a,b) = T^{-1}(p,q)$ —— ①

Now, $T(a,b) = (p,q)$

$\Rightarrow (a+b, a) = p, q$

$\Rightarrow a+b = p, \quad a = q$

$\Rightarrow b = p-q, \quad a = q$

$\therefore$ ① $\Rightarrow T^{-1}(p,q) = (q, p-q)$ which is the required rule of T^{-1}

Q) If, $T: R^2(R) \to R^2(R)$ is linear operator defined by $T(x,y) = (x-y, y)$ then $T^2(x,y) = ?$

Soln Given,

$T: R^2(R) \to R^2(R)$ by

$T(x,y) = (x-y, y)$

Now, $T^2(x,y) = T(T(x,y))$

$= T(x-y, y)$

$= (x-y-y, y)$

$= (x-2y, y)$

Q) Show that identity operator on a vector space is always invertible.

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*) Co-ordinates:

Let, $V(F)$ be a finite dimensional vector space. Let,

$B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for.

Let, $\alpha \in V$. Then there exist a unique ~~and~~ ~~sca~~ 'n' nos. scalars $(a_1, a_2, \ldots a_n)$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_m = \sum_{i=1}^{n} a_i\alpha_i$$

The n-tuple $(a_1, a_2, \ldots, a_n)$ is called co-ordinates of α relative to the ordered basis B. The $n \times 1$ matrix

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

is called co-ordinate matrix of ~~the~~ α relative to B.

2) Find the co-ordinates of the vector (a, b, c) w.r.t the basis $B = \{(1,0,0); (1,1,0), (1,1,1)\}$ of R^3

Solⁿ: Given,

$$S = \{(1,0,0),(1,1,0),(1,1,1)\}$$

Let, $(a,b,c) \in R^3$ and p, q, r be any three scalar

Now, $(a,b,c) = p(1,0,0) + q(1,1,0) + r(1,1,1)$

$\Rightarrow (a,b,c) = (p,0,0) + (q,q,0) + (r,r,r)$

$\Rightarrow (a,b,c) = (p+q+r, q+r, r)$

$\Rightarrow a = p+q+r, \quad b = q+r, \quad c = r$

$\Rightarrow a = p+b-c+c \quad , \quad \Rightarrow b = q+c$

$\Rightarrow a-b = p \quad , \quad \Rightarrow b-c = q$

Hence co-ordinate of the vector (a,b,c) is

$(p, q, r) = (a-b, b-c, c)$

Q) Find the co-ordinates of the vector $(2,1,-6)$ of R^3 relative to the basis $\alpha_1=(1,1,2)$, $\alpha_2=(3,-1,0)$, $\alpha_3=(2,0,-1)$

$\frac{15}{8}, \frac{7}{8}, \frac{17}{3}$

Sol^n: Given,

$\alpha_1 = (1,1,2)$

$\alpha_2 = (3,-1,0)$

$\alpha_3 = (2,0,-1)$

~~Let~~, Given,
$(2,1,-6) \in R^3$ and let p, q, r be any three scaler

Now, $(2,1,-6) = p(1,1,2) + q(3,-1,0) + r(2,0,-1)$

$(2,1,-6) = p, p, 2p + 3q, -q, + 2r, -r$

$(2,1,-6) = p+3q+2r,\ p-q,\ 2p-r$

$2 = p+3q+2r, \quad 1 = p-q, \quad -6 = 2p-r$

$2 = 1+q+3q+12+4p, \quad p = 1+q, \quad 6+2p = r$

$-11 = 4q+4(1+q) \qquad p = 1-\frac{7}{8} \qquad 6+2(-\frac{1}{6}) = r$

$-11 = 4q+4+4q \qquad = \frac{8-7}{8} \qquad 6-\frac{2}{6} = r$

$-7 = 8q \qquad = \frac{-1}{8} \qquad \Rightarrow \frac{36-2}{6} = r$

$-\frac{7}{8} = q \qquad \Rightarrow \frac{34}{6} = r$

Notes given by Ragib sir