* Limit point of a set :-

Let E be a subset of $\mathbb{R}$. and $x \in \mathbb{R}$. be any point (the point x may or may not belong to E). We say x, is a limit point (or cluster point or accumulation point) of E. if every neighbourhood of x contains atleast one point of E other than the point x. itself.

Or If every deleted neighbourhood of x contains atleast one point of E.

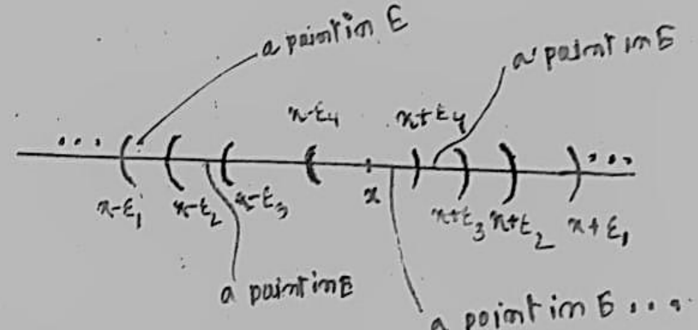
Or for every $\varepsilon > 0$ (however small)

$$(x-\varepsilon, x+\varepsilon) \setminus \{x\}$$ contains atleast one point of E.

Or for every $\varepsilon > 0$ (however small)

$$(x-\varepsilon, x+\varepsilon) \setminus \{x\} \cap E \neq \phi .$$

Illustration:-



Whatever the neighbourhood of x if it contains atleast one point of E other than x then x is a limit point of E.

Examples:

(1) Every point of the open interval (a,b) and the points a and b are limit points of (a,b).

Proof: let x be any point in (a,b), and let $\epsilon > 0$ be any real number. consider the neighbourhood $(x-\epsilon, x+\epsilon)$.

Case I: If $x-\epsilon > a$ or $x+\epsilon < b$.
then $(x-\epsilon, x+\epsilon)$ contains atleast one point of E.

Case II: If not then $(a,b) \subseteq (x-\epsilon, x+\epsilon)$

$\therefore (x-\epsilon, x+\epsilon) \setminus \{x\} \cap (a,b) \neq \phi$.

$\therefore$ x is limit point of (a,b) and since x is any orbitrary point therefore every point $x \in (a,b)$ is a limit point of (a,b).

Now consider the point a and any neighbourhood $(a-\epsilon, a+\epsilon)$ of a.

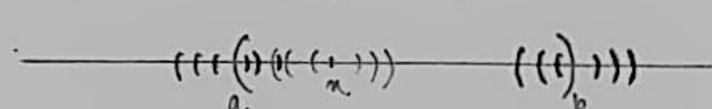
$\because a+\epsilon > a$ for every $\epsilon > 0$

$\therefore (a-\epsilon, a+\epsilon) \setminus \{a\} \cap (a,b) \neq \phi$.

$\therefore$ a is a limit point of a.

Similar is the case with the point b.

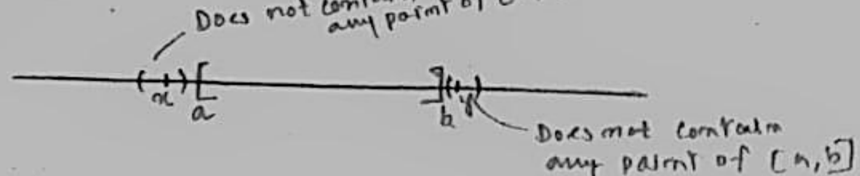
Illustration:-



(2) Every point of $[a,b]$ is a limit point of $[a,b]$.

Note: No point outside $[a,b]$ is a limit point of $[a,b]$.

Illustration:-



(3) No point of $\mathbb{N}$ is an accumulation point of $\mathbb{N}$.

<u>Proof:</u> let $n \in \mathbb{N}$ be any natural number.

consider the interval $(n - 1/2, n + 1/2)$.

the interval does not contain any natural number other than n.

$\therefore$ n is not a limit point of $\mathbb{N}$.

$\because$ n is any arbitrary point of $\mathbb{N}$, therefore—
no point in $\mathbb{N}$ is a limit point of $\mathbb{N}$.

(4) Similarly no point of $\mathbb{Z}$, $E = \{2n ; n \in \mathbb{N}\}$ is a limit point of the corresponding sets.

(5) Every point of $\mathbb{R}$ is a limit point.

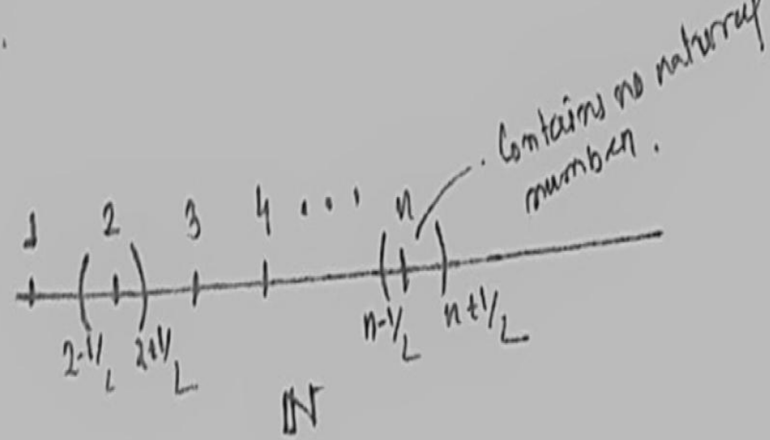
<u>proof!</u> Consider any point $x \in \mathbb{R}$, and any neighbourhood $(x-\varepsilon, x+\varepsilon)$ of x,

then $(x-\varepsilon, x+\varepsilon) \setminus \{x\} \cap \mathbb{R} = (x-\varepsilon, x+\varepsilon) \setminus \{x\}$

Which contains atleast one point of $\mathbb{R}$ for every $\varepsilon > 0$.

Illustration:

(3)



(5)