Interpolation

Interpolation is the process of estimating the value of function for any intermediate value of the variable with the help of its given set of values.

Let us assume that the function $y = f(x)$ is known for certain values of x say a for $x_0, x_1, x_2, \ldots x_n$

As $f(x_0), f(x_1), \ldots f(x_n)$ The process of finding the value of $f(x)$ corresponding to $x = x_i$; where $x_0 < x_i < x_n$. with the help of given data is called interpolation.

Forward difference operator

Let $y = f(x)$ be the function attain the values of $y_0, y_1, y_2, \ldots y_n$ corresponding to the equidistance values $x, x+h, x+2h, \ldots x+nh$ of the arguments x then forward difference operator is denoted by Δ and is defined as

$$\Delta f(x) = f(x+h) - f(x) \quad \text{———} \quad (1)$$

or $\Delta y_0 = y_1 - y_0$ 2) $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$= \Delta\Delta y_1 - \Delta\Delta y_0$$
$$= \Delta(y_2 - y_1) - \Delta(y_1 - y_0)$$
$$= \Delta y_2 - \Delta y_1 - \Delta y_1 + \Delta y_0$$

$= \Delta y_2 - 2\Delta y_1 + \Delta y_0$

$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0)$

$= y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0$

$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

continuing this process we get nth forward difference is given by

$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$$

for $\boxed{\Delta^n y_0 = \Delta^{n-1} y_1 - \Delta^{n-1} y_0}$

Backward difference

Backward difference is just parallel to the concept of forward difference. It is denoted by ∇ and defined as

$$\nabla f(x) = f(x) - f(x-h)$$

or $\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$

$\nabla y_3 = y_3 - y_2$

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continuing this process we get

$$\boxed{\nabla y_n = y_n - y_{n-1}}$$

The shift operator

The shift operator which is denoted by E and defined as $\boxed{Ef(x) = f(x+h)}$

where h is the increment in the value of x.

Slly, $E^2 f(x) = EEf(x) = Ef(x+h) = f(x+2h)$

and $E^3 f(x) = f(x+3h)$

In general $\boxed{E^n f(x) = f(x+nh)}$

Q) Prove that the following relation between the operator

1) $\Delta = E - 1$

2) $\nabla = 1 - E^{-1}$

3) $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

4) $\mu = \frac{1}{2}\left\{E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right\}$

5) $\Delta = E\nabla = \Delta\nabla E = \delta E^{\frac{1}{2}}$

6) $E = e^{hD}$

7) $(1+\Delta)(1-\nabla) = 1$

1) **Proof:** We know that,

~~forward~~ L.H.S $\Delta f(x) = f(x+h) - f(x)$

$= Ef(x) - f(x)$ — (1)

$\Rightarrow \cancel{Ef(x) =}$

R.H.S.

$$(E-1)f(x) = Ef(x) - f(x) \quad \text{--- (ii)}$$

From (i) and (ii)

L.H.S = R.H.S

Hence proved

(2) proof

$$\nabla = 1 - E^{-1}$$

L.H.S.:

$$\nabla f(x) = f(x) - f(x-h)$$

RHS.

$$(1-E^{-1})f(x) = f(x) - E^{-1}f(x)$$
$$= f(x) - f(x-h)$$

$\therefore$ L.H.S = R.H.S

7) $(1+\Delta)(1-\nabla) = 1$

Note:
$$\Delta f(x) = f(x+h) - f(x)$$

Proof: $(1+\Delta)(1-\nabla)f(x)$

$$= (1+\Delta)(f(x) - \nabla f(x))$$
$$= (1+\Delta)(f(x) - (f(x) - f(x-h)))$$
$$= (1+\Delta)(f(x) - f(x) + f(x-h))$$
$$= (1+\Delta)(f(x-h))$$
$$= f(x-h) + \Delta f(x-h)$$
$$= \cancel{f(x-h)} + f(x-h+h) - \cancel{f(x-h)}$$
$$= f(x)$$

$\therefore \quad (1+\Delta)(1-\nabla) f(x) = f(x)$

$\Rightarrow \quad (1+\Delta)(1-\nabla) = 1.$

Proved

Central difference operator

The central difference operator denoted by δ and defined as

$$\boxed{\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}$$

Averaging operator

The averaging operator which is denoted by μ and defined as

$$\boxed{\mu f(x) = \frac{f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)}{2}}$$

Q) Relation betn central diffn operator and shift operator

$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Proof We know that

$$\delta f(x) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$$\delta f(x) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

(4) $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{\frac{1}{2}}\right)$

Proof

We know that

$$\mu f(x) = \frac{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)}{2}$$

$$= \frac{E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)}{2}$$

$$\mu f(x) = \frac{\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) f(x)}{2}$$

$$\Rightarrow \mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

Construct a forward different table for the following values:-

x : 0 5 10 15 20 25

$f(x)$: 7 11 14 18 24 32

Sol^n

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	7					
		4				
5	11		-1			
		3		2		
10	14		1		-1	
		4		1		0
15	18		2		-1	
		6		0		
20	24		2			
		8				
25	32					

Q) Estimate the missing term in the following table

x : 0 1 2 3 4

$y = f(x)$: 1 3 9 ? 81.

Sol^n Here, we are given ~~four~~ five values of x

$\therefore \Delta^4 f(x) = 0 \ \forall x$

We know that,

$$\Delta = E - 1$$

$$\Rightarrow \Delta^4 = (E-1)^4$$

Now, $\Delta^4 f(x) = 0$

$$\Rightarrow (E-1)^4 f(x) = 0$$

$$\Rightarrow (E^2 - 2E + 1)^2 f(x) = 0$$

$$\Rightarrow (E^2 - 2E + 1)(E^2 - 2E + 1) f(x) = 0$$

$$\Rightarrow (E$$

$\Rightarrow (E^4 - 2E^3 + E^2 - 2E^3 + 4E^2 - 2E + E^2 - 2E + 1) f(x) = 0$

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) f(x) = 0$

$\Rightarrow E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4E f(x) + f(x) = 0$

$\Rightarrow \cancel{E^4} f(x+4h) - 4\cancel{E^3} f(x+3h) + 6\cancel{E^2} f(x+2h) - 4\cancel{E} f(x+h) + f(x) = 0$

$\Rightarrow \cancel{E^4} f(x+4) - 4\cancel{E^3} f(x+3) + 6\cancel{E^2} f(x+2) - 4\cancel{E} f(x+1) + f(x) = 0$ $[\because h = 1]$

When, $x = 0$

$\Rightarrow \cancel{E^4} f(4) - 4\cancel{E^3} f(3) + 6 f(2) - 4 f(1) + f(0) = 0$

$\Rightarrow 81 - 4 f(3) + 6 \cdot 9 - 4 \cdot 3 + 1 = 0$

$\Rightarrow 4 f(3) = 124$

$\Rightarrow f(3) = 31$

Note:- $E f(x) = f(x+h)$

$\Delta f(x) = f(x+h) - f(x)$

$\Delta \sin^{-1} x$:

$= \sin^{-1}(x+h) - \sin^{-1} x$

you can proceed in this way too

Q) Find, $\Delta \tan^{-1} x$

$= (E - 1) \tan^{-1} x$

$= E \tan^{-1} x - \tan^{-1} x$

$= \tan^{-1}(x+h) - \tan^{-1} x$.

$= \tan^{-1} \left(\frac{x+h-x}{1+(x+h) \cdot x} \right)$

$= \tan^{-1} \left(\frac{h}{1+x^2+xh} \right)$

Solved.

Newton's ~~formula~~ forward interpolation formula

Let the function $y = f(x)$ take the values $y_0, y_1, y_2 \ldots$ corresponding to the values $x_0, x_0+h, x_0+2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + uh$, where u is any real number.

For any real number u, we have defined E such that

$$E^u f(x) = f(x+uh)$$

$$y_u = f(x_0+uh) = E^u f(x_0) = (1+\Delta)^u y_0$$

$$= \left\{1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \cdots\right\} y_0$$

[using Binomial theorem]

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

It is called the Newton's Gregory forward interpolation formula.

Newton's backward interpolation formula

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0+h, x_0+\ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + uh$. Where u is any real number.

Then we have

$$y_n = f(x_n + up) = E^u f(x_n) = (1-\nabla)^{-u} y_n$$

$$= \left[1 + u\nabla + \frac{u(u+1)}{2!}\nabla^2 + \frac{u(u+1)(u+2)}{3!}\nabla^3 + \ldots\right] y_n$$

[using Binomial theorem]

i.e. $$y_u = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots$$

It is called Newton's Gregory backward interpolation formula.

Observation

Q) The table gives

x = hight : 100 150 200 250 300 350 400

y = distance : 10.63 13.03 15.04 16.81 18.42 19.90 21.27

Find the value of y when $x = 218$ and $x = 410$

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
100	10.63						
		2.4					
150	13.03		-0.39				
		2.1		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

If we $x_0 = 200$ then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$,
$\Delta^3 y_0 = 0.03$, $\Delta^4 y_0 = -0.01$

Since, $x = 218$ and $h = 50$

$$\therefore u = \frac{x - x_0}{h}$$

$$= \frac{218 - 200}{50}$$

$$= 0.36$$

Now, By using Newton forward interpolation formula we have

$$y_{218} = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 15.04 + 0.36(1.77) + \frac{0.36(0.36-1)}{2}(-0.16) + \frac{0.36(0.36-1)(0.36-2)}{6}(0.03)$$

$$= 15.04 + 0.637 + 0.018 + 0.001 + \cdots$$

$= 15.696$

For $x = 410$ is near the end of the table, we use Newton's backward interpolation formula

$\therefore$ Taking $x_n = 400$, $p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward difference

$y_0 = 21.27$, $\nabla y_0 = 1.37$, $\nabla^2 y_0 = -0.11$, $\nabla^3 y_0 = 0.02$, $\nabla^4 y = -0.01$, $\nabla^5 y = 0.04$

$\nabla^6 y = 0.02$

$\therefore$ Newton's backward formula we have

$$y_{410} = y_{400} + u \nabla y_0 + \frac{u(u+1)}{2!} \nabla^2 y_0 + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_0 + \cdots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \cdots$$

$$= 21.53$$

Interpollation with unequal intervals:

Lagrange formula for unequal interval

If $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n \quad (1)$$

This is known as Lagrange's interpolation formula for unequal intervals.

Q) Given the value

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evalute $f(9)$ using lagrange formula

② Newton's divides difference formula

Sol: Here,

$x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$y_0 = 150$, $y_1 = 392$, $y_2 = 1452$, $y_3 = 2366$, $y_4 = 5202$

We Know that,

By Lagrange formula,

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Putting ~~Let~~, $x = 9$

$$f(9) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

$$= \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150) + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452) + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202$$

$$= \cancel{64}\cdot\frac{1800}{\cancel{28}} + \cancel{2400}\ 1200\ 50$$

$$9600\quad 4800$$

$$= \frac{-19200}{\cancel{1152}} + \frac{1003252}{40} + \frac{371712}{288} - \frac{302848}{384} + \frac{332328}{2880}$$

$$\cancel{576}\quad \cancel{288}\quad \cancel{144}\quad \cancel{72}\ 8$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{\cancel{13847}}{\cancel{120}}\ \frac{578}{5}$$

$$= \frac{\cancel{-200+25088+154880-94640+13847}}{120}$$

$$= \cancel{810}\ \frac{-250+3136+19360-11830+1734}{15}$$

$$= 810$$

Notes given by Rajib sir