

Chapter 1

Vector Analysis

Problem 1.1

(a) From the diagram, $|\mathbf{B} + \mathbf{C}| \cos \theta_3 = |\mathbf{B}| \cos \theta_1 + |\mathbf{C}| \cos \theta_2$. Multiply by $|\mathbf{A}|$.

$$|\mathbf{A}||\mathbf{B} + \mathbf{C}| \cos \theta_3 = |\mathbf{A}||\mathbf{B}| \cos \theta_1 + |\mathbf{A}||\mathbf{C}| \cos \theta_2.$$

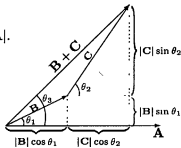
So: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$. (Dot product is distributive.)

Similarly: $|\mathbf{B} + \mathbf{C}| \sin \theta_3 = |\mathbf{B}| \sin \theta_1 + |\mathbf{C}| \sin \theta_2$. Multiply by $|\mathbf{A}| \hat{\mathbf{n}}$.

$$|\mathbf{A}||\mathbf{B} + \mathbf{C}| \sin \theta_3 \hat{\mathbf{n}} = |\mathbf{A}||\mathbf{B}| \sin \theta_1 \hat{\mathbf{n}} + |\mathbf{A}||\mathbf{C}| \sin \theta_2 \hat{\mathbf{n}}.$$

If $\hat{\mathbf{n}}$ is the unit vector pointing out of the page, it follows that

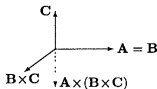
$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}). \quad (\text{Cross product is distributive.})$$



(b) For the general case, see G. E. Hay's *Vector and Tensor Analysis*, Chapter 1, Section 7 (dot product) and Section 8 (cross product).

Problem 1.2

The triple cross-product is *not* in general associative. For example, suppose $\mathbf{A} = \mathbf{B}$ and \mathbf{C} is perpendicular to \mathbf{A} , as in the diagram. Then $(\mathbf{B} \times \mathbf{C})$ points out-of-the-page, and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ points down, and has magnitude ABC . But $(\mathbf{A} \times \mathbf{B}) = 0$, so $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = 0 \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

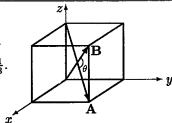


Problem 1.3

$$\mathbf{A} = +1\hat{x} + 1\hat{y} - 1\hat{z}; \quad \mathbf{A} = \sqrt{3}; \quad \mathbf{B} = 1\hat{x} + 1\hat{y} + 1\hat{z}; \quad B = \sqrt{3}.$$

$$\mathbf{A} \cdot \mathbf{B} = +1 + 1 - 1 = 1 = AB \cos \theta = \sqrt{3}\sqrt{3} \cos \theta \Rightarrow \cos \theta = \frac{1}{3}.$$

$$\theta = \cos^{-1} \left(\frac{1}{3} \right) \approx 70.5288^\circ$$



Problem 1.4

The cross-product of any two vectors in the plane will give a vector perpendicular to the plane. For example, we might pick the base (\mathbf{A}) and the left side (\mathbf{B}):

$$\mathbf{A} = -1\hat{x} + 2\hat{y} + 0\hat{z}; \quad \mathbf{B} = -1\hat{x} + 0\hat{y} + 3\hat{z}.$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{x} + 3\hat{y} + 2\hat{z}.$$

This has the right *direction*, but the wrong *magnitude*. To make a *unit* vector out of it, simply divide by its length:

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{36 + 9 + 4} = 7. \quad \hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \left[\frac{6}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{2}{7}\hat{z} \right].$$

Problem 1.5

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ (B_y C_z - B_z C_y) & (B_z C_x - B_x C_z) & (B_x C_y - B_y C_x) \end{vmatrix}$$

(I'll just check the x-component; the others go the same way.)

$$= \hat{x}[A_y(B_z C_y - B_y C_z) - A_z(B_z C_x - B_x C_z)] + \hat{y}(\dots) + \hat{z}(\dots)$$

$$= \hat{x}(A_y B_z C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z) + \hat{y}(\dots) + \hat{z}(\dots).$$

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = [B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z B_z)]\hat{x} + (\dots)\hat{y} + (\dots)\hat{z}$$

= $\hat{x}(A_y B_z C_y + A_z B_x C_z - A_y B_y C_x - A_z B_z C_x) + \hat{y}(\dots) + \hat{z}(\dots)$. They agree.

Problem 1.6

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) - \mathbf{B}(\mathbf{C} \cdot \mathbf{A}) = 0.$$

So: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) - (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

If this is zero, then either \mathbf{A} is parallel to \mathbf{C} (including the case in which they point in *opposite* directions, or one is zero), or else $\mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{A} = 0$, in which case \mathbf{B} is perpendicular to \mathbf{A} and \mathbf{C} (including the case $\mathbf{B} = 0$).

Conclusion: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \iff$ either \mathbf{A} is parallel to \mathbf{C} , or \mathbf{B} is perpendicular to \mathbf{A} and \mathbf{C} .

Problem 1.7

$$\mathbf{a} = (4\hat{x} + 6\hat{y} + 8\hat{z}) - (2\hat{x} + 8\hat{y} + 7\hat{z}) = \boxed{2\hat{x} - 2\hat{y} + \hat{z}}$$

$$a = \sqrt{4 + 4 + 1} = \boxed{3}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{a} = \left[\frac{2}{3}\hat{x} - \frac{2}{3}\hat{y} + \frac{1}{3}\hat{z} \right]$$

Problem 1.8

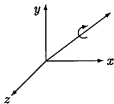
$$\begin{aligned} \text{(a)} \quad \bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z &= (\cos \phi A_y + \sin \phi A_z)(\cos \phi B_y + \sin \phi B_z) + (-\sin \phi A_y + \cos \phi A_z)(-\sin \phi B_y + \cos \phi B_z) \\ &= \cos^2 \phi A_y B_y + \sin \phi \cos \phi (A_y B_z + A_z B_y) + \sin^2 \phi A_z B_z + \sin^2 \phi A_y B_y - \sin \phi \cos \phi (A_y B_z + A_z B_y) + \\ &\quad \cos^2 \phi A_z B_z \\ &= (\cos^2 \phi + \sin^2 \phi) A_y B_y + (\sin^2 \phi + \cos^2 \phi) A_z B_z = A_y B_y + A_z B_z. \quad \checkmark \end{aligned}$$

$$\text{(b)} \quad (\bar{A}_x)^2 + (\bar{A}_y)^2 + (\bar{A}_z)^2 = \sum_{i=1}^3 \bar{A}_i \bar{A}_i = \sum_{i=1}^3 (\sum_{j=1}^3 R_{ij} A_j) (\sum_{k=1}^3 R_{ik} A_k) = \sum_{j,k} (\sum_i R_{i,j} R_{i,k}) A_j A_k.$$

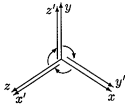
$$\text{This equals } A_x^2 + A_y^2 + A_z^2 \text{ provided } \sum_{i=1}^3 R_{i,j} R_{i,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Moreover, if R is to preserve lengths for *all* vectors \mathbf{A} , then this condition is not only *sufficient* but also *necessary*. For suppose $\mathbf{A} = (1, 0, 0)$. Then $\sum_{j,k} (\sum_i R_{i,j} R_{i,k}) A_j A_k = \sum_i R_{i,1} R_{i,1}$, and this must equal 1 (since we want $\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = 1$). Likewise, $\sum_{i=1}^3 R_{i,2} R_{i,2} = \sum_{i=1}^3 R_{i,3} R_{i,3} = 1$. To check the case $j \neq k$, choose $\mathbf{A} = (1, 1, 0)$. Then we want $2 = \sum_{j,k} (\sum_i R_{i,j} R_{i,k}) A_j A_k = \sum_i R_{i,1} R_{i,1} + \sum_i R_{i,2} R_{i,2} + \sum_i R_{i,1} R_{i,2} + \sum_i R_{i,2} R_{i,1}$. But we already know that the first two sums are both 1; the third and fourth are *equal*, so $\sum_i R_{i,1} R_{i,2} = \sum_i R_{i,2} R_{i,1} = 0$, and so on for other unequal combinations of j, k . \checkmark In matrix notation: $\bar{R}R = 1$, where \bar{R} is the *transpose* of R .

Problem 1.9



Looking down the axis:



A 120° rotation carries the z axis into the y ($= \bar{z}$) axis, y into x ($= \bar{y}$), and x into z ($= \bar{x}$). So $\bar{A}_z = A_z$, $\bar{A}_y = A_x$, $\bar{A}_x = A_y$.

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Problem 1.10

(a) **No change.** ($\bar{A}_x = A_x$, $\bar{A}_y = A_y$, $\bar{A}_z = A_z$)(b) **A \rightarrow $-A$,** in the sense ($\bar{A}_x = -A_x$, $\bar{A}_y = -A_y$, $\bar{A}_z = -A_z$)

(c) $(\mathbf{A} \times \mathbf{B}) \rightarrow (-\mathbf{A}) \times (-\mathbf{B}) = (\mathbf{A} \times \mathbf{B})$. That is, if $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, **C \rightarrow C**. No minus sign, in contrast to behavior of an "ordinary" vector, as given by (b). If \mathbf{A} and \mathbf{B} are *pseudovectors*, then $(\mathbf{A} \times \mathbf{B}) \rightarrow (\mathbf{A}) \times (\mathbf{B}) = (\mathbf{A} \times \mathbf{B})$. So the cross-product of two pseudovectors is again a *pseudovector*. In the cross-product of a vector and a pseudovector, one changes sign, the other doesn't, and therefore the cross-product is itself a *vector*. *Angular momentum* ($\mathbf{L} = \mathbf{r} \times \mathbf{p}$) and *torque* ($\mathbf{N} = \mathbf{r} \times \mathbf{F}$) are pseudovectors.

(d) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \rightarrow (-\mathbf{A}) \cdot ((-\mathbf{B}) \times (-\mathbf{C})) = -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. So, if $a = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, then **a \rightarrow $-a$** ; a pseudoscalar *changes sign* under inversion of coordinates.

Problem 1.11

$$(a) \nabla f = 2x \hat{x} + 3y^2 \hat{y} + 4z^3 \hat{z}$$

$$(b) \nabla f = 2xy^3z^4 \hat{x} + 3x^2y^2z^4 \hat{y} + 4x^2y^3z^3 \hat{z}$$

$$(c) \nabla f = e^x \sin y \ln z \hat{x} + e^x \cos y \ln z \hat{y} + e^x \sin y (1/z) \hat{z}$$

Problem 1.12

(a) $\nabla h = 10[(2y - 6x - 18) \hat{x} + (2x - 8y + 28) \hat{y}]$. $\nabla h = 0$ at summit, so

$$\left. \begin{aligned} 2y - 6x - 18 &= 0 \\ 2x - 8y + 28 &= 0 \implies 6x - 24y + 84 = 0 \end{aligned} \right\} 2y - 18 - 24y + 84 = 0.$$

$$22y = 66 \implies y = 3 \implies 2x - 24 + 28 = 0 \implies x = -2.$$

Top is **3 miles north, 2 miles west, of South Hadley.**

(b) Putting in $x = -2$, $y = 3$:

$$h = 10(-12 - 12 - 36 + 36 + 84 + 12) = \mathbf{720 \text{ ft.}}$$

(c) Putting in $x = 1$, $y = 1$: $\nabla h = 10[(2 - 6 - 18) \hat{x} + (2 - 8 + 28) \hat{y}] = 10(-22 \hat{x} + 22 \hat{y}) = 220(-\hat{x} + \hat{y})$.

$$|\nabla h| = 220\sqrt{2} \approx \mathbf{311 \text{ ft/mile}}; \text{ direction: } \mathbf{\text{northwest.}}$$

Problem 1.13

$$\mathbf{r} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}; \quad r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

$$(a) \nabla(r^2) = \frac{\partial}{\partial x}[(x - x')^2 + (y - y')^2 + (z - z')^2]\hat{x} + \frac{\partial}{\partial y}(\dots)\hat{y} + \frac{\partial}{\partial z}(\dots)\hat{z} = 2(x - x')\hat{x} + 2(y - y')\hat{y} + 2(z - z')\hat{z} = 2\mathbf{r}.$$

$$(b) \nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}[(x - x')^2 + (y - y')^2 + (z - z')^2]^{-\frac{1}{2}}\hat{x} + \frac{\partial}{\partial y}(\dots)^{-\frac{1}{2}}\hat{y} + \frac{\partial}{\partial z}(\dots)^{-\frac{1}{2}}\hat{z}$$

$$= -\frac{1}{2}(\dots)^{-\frac{3}{2}}2(x - x')\hat{x} - \frac{1}{2}(\dots)^{-\frac{3}{2}}2(y - y')\hat{y} - \frac{1}{2}(\dots)^{-\frac{3}{2}}2(z - z')\hat{z}$$

$$= -(\dots)^{-\frac{3}{2}}[(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}] = -(1/r^3)\mathbf{r} = -(1/r^2)\hat{\mathbf{r}}.$$

$$(c) \frac{\partial}{\partial x}(r^n) = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\left(\frac{1}{2}2\hat{x}\right) = nr^{n-1}\hat{x}, \text{ so } \boxed{\nabla(r^n) = nr^{n-1}\hat{\mathbf{r}}}.$$

Problem 1.14

$$\bar{y} = +y \cos \phi + z \sin \phi; \text{ multiply by } \sin \phi: \bar{y} \sin \phi = +y \sin \phi \cos \phi + z \sin^2 \phi.$$

$$\bar{z} = -y \sin \phi + z \cos \phi; \text{ multiply by } \cos \phi: \bar{z} \cos \phi = -y \sin \phi \cos \phi + z \cos^2 \phi.$$

$$\text{Add: } \bar{y} \sin \phi + \bar{z} \cos \phi = z(\sin^2 \phi + \cos^2 \phi) = z. \text{ Likewise, } \bar{y} \cos \phi - \bar{z} \sin \phi = y.$$

$$\text{So } \frac{\partial \bar{y}}{\partial y} = \cos \phi; \frac{\partial \bar{y}}{\partial z} = -\sin \phi; \frac{\partial \bar{z}}{\partial y} = \sin \phi; \frac{\partial \bar{z}}{\partial z} = \cos \phi. \text{ Therefore}$$

$$\left. \begin{aligned} \frac{\partial(\bar{y}f)}{\partial y} &= \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} = +\cos \phi(\nabla f)_y + \sin \phi(\nabla f)_z \\ \frac{\partial(\bar{y}f)}{\partial z} &= \frac{\partial f}{\partial z} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial z} = -\sin \phi(\nabla f)_y + \cos \phi(\nabla f)_z \end{aligned} \right\} \text{ So } \nabla f \text{ transforms as a vector. } \quad \text{qed}$$

Problem 1.15

$$(a) \nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) = 2x + 0 - 2x = 0.$$

$$(b) \nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3xz) = y + 2x + 3x.$$

$$(c) \nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz) = 0 + (2x) + (2y) = 2(x + y).$$

Problem 1.16

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right) = \frac{\partial}{\partial x}\left[x(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial y}\left[y(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial z}\left[z(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right]$$

$$= (-\frac{3}{2} + x(-3/2)(\dots)^{-\frac{5}{2}})2x + (-\frac{3}{2} + y(-3/2)(\dots)^{-\frac{5}{2}})2y + (-\frac{3}{2} + z(-3/2)(\dots)^{-\frac{5}{2}})2z$$

$$= 3r^{-3} - 3r^{-5}(x^2 + y^2 + z^2) = 3r^{-3} - 3r^{-3} = 0.$$

This conclusion is surprising, because, from the diagram, this vector field is obviously diverging away from the origin. How, then, can $\nabla \cdot \mathbf{v} = 0$? The answer is that $\nabla \cdot \mathbf{v} = 0$ everywhere *except* at the origin, but at the origin our calculation is no good, since $r = 0$, and the expression for \mathbf{v} blows up. In fact, $\nabla \cdot \mathbf{v}$ is *infinite* at that one point, and zero elsewhere, as we shall see in Sect. 1.5.

Problem 1.17

$$\bar{y}_y = \cos \phi v_y + \sin \phi v_z; \quad \bar{v}_z = -\sin \phi v_y + \cos \phi v_z.$$

$$\frac{\partial \bar{y}_y}{\partial y} = \frac{\partial v_y}{\partial y} \cos \phi + \frac{\partial v_z}{\partial y} \sin \phi = \left(\frac{\partial v_x}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial v_x}{\partial z} \frac{\partial z}{\partial y}\right) \cos \phi + \left(\frac{\partial v_x}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial v_x}{\partial z} \frac{\partial z}{\partial y}\right) \sin \phi. \text{ Use result in Prob. 1.14:}$$

$$= \left(\frac{\partial v_x}{\partial y} \cos \phi + \frac{\partial v_x}{\partial z} \sin \phi\right) \cos \phi + \left(\frac{\partial v_x}{\partial y} \cos \phi + \frac{\partial v_x}{\partial z} \sin \phi\right) \sin \phi.$$

$$\frac{\partial \bar{v}_z}{\partial z} = -\frac{\partial v_y}{\partial z} \sin \phi + \frac{\partial v_z}{\partial z} \cos \phi = -\left(\frac{\partial v_x}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial v_x}{\partial z} \frac{\partial z}{\partial z}\right) \sin \phi + \left(\frac{\partial v_x}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial v_x}{\partial z} \frac{\partial z}{\partial z}\right) \cos \phi$$

$$= -\left(\frac{\partial v_x}{\partial y} \sin \phi + \frac{\partial v_x}{\partial z} \cos \phi\right) \sin \phi + \left(-\frac{\partial v_x}{\partial y} \sin \phi + \frac{\partial v_x}{\partial z} \cos \phi\right) \cos \phi. \text{ So}$$

$$\frac{\partial \bar{y}_y}{\partial y} + \frac{\partial \bar{v}_z}{\partial z} = \frac{\partial v_x}{\partial y} \cos^2 \phi + \frac{\partial v_x}{\partial z} \sin \phi \cos \phi + \frac{\partial v_x}{\partial y} \sin \phi \cos \phi + \frac{\partial v_x}{\partial z} \sin^2 \phi + \frac{\partial v_x}{\partial y} \sin^2 \phi - \frac{\partial v_x}{\partial z} \sin \phi \cos \phi$$

$$= \frac{\partial v_x}{\partial y} (\cos^2 \phi + \sin^2 \phi) + \frac{\partial v_x}{\partial z} (\sin^2 \phi + \cos^2 \phi) = \frac{\partial v_x}{\partial y} + \frac{\partial v_x}{\partial z} \cdot \checkmark$$

Problem 1.18

$$(a) \nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = \hat{x}(0 - 6xz) + \hat{y}(0 + 2z) + \hat{z}(3z^2 - 0) = \boxed{-6xz \hat{x} + 2z \hat{y} + 3z^2 \hat{z}}$$

$$(b) \nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = \hat{x}(0 - 2y) + \hat{y}(0 - 3z) + \hat{z}(0 - x) = \boxed{-2y \hat{x} - 3z \hat{y} - x \hat{z}}$$

$$(c) \nabla \times \mathbf{v}_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix} = \hat{x}(2z - 2z) + \hat{y}(0 - 0) + \hat{z}(2y - 2y) = \boxed{0}$$

Problem 1.19

$\mathbf{v} = y \hat{x} + x \hat{y}$; or $\mathbf{v} = yz \hat{x} + xz \hat{y} + xy \hat{z}$; or $\mathbf{v} = (3x^2z - z^3) \hat{x} + 3\hat{y} + (x^3 - 3xz^2) \hat{z}$;
or $\mathbf{v} = (\sin x)(\cosh y) \hat{x} - (\cos x)(\sinh y) \hat{y}$; etc.

Problem 1.20

$$(i) \nabla(fg) = \frac{\partial(fg)}{\partial x} \hat{x} + \frac{\partial(fg)}{\partial y} \hat{y} + \frac{\partial(fg)}{\partial z} \hat{z} = \left(f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right) \hat{x} + \left(f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y} \right) \hat{y} + \left(f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z} \right) \hat{z} \\ = f \left(\frac{\partial g}{\partial x} \hat{x} + \frac{\partial g}{\partial y} \hat{y} + \frac{\partial g}{\partial z} \hat{z} \right) + g \left(\frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \right) = f(\nabla g) + g(\nabla f). \quad \text{qed}$$

$$(iv) \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \frac{\partial}{\partial x} (A_y B_z - A_z B_y) + \frac{\partial}{\partial y} (A_z B_x - A_x B_z) + \frac{\partial}{\partial z} (A_x B_y - A_y B_x) \\ = A_y \frac{\partial B_z}{\partial x} + B_z \frac{\partial A_y}{\partial x} - A_z \frac{\partial B_y}{\partial x} - B_y \frac{\partial A_z}{\partial x} + A_z \frac{\partial B_x}{\partial y} + B_x \frac{\partial A_z}{\partial y} - A_x \frac{\partial B_z}{\partial y} - B_z \frac{\partial A_x}{\partial y} \\ + A_x \frac{\partial B_y}{\partial z} + B_y \frac{\partial A_x}{\partial z} - A_y \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_y}{\partial z} \\ = B_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + B_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + B_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - A_x \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \\ - A_y \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) - A_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \quad \text{qed}$$

$$(v) \nabla \times (f\mathbf{A}) = \left(\frac{\partial(fA_x)}{\partial y} - \frac{\partial(fA_y)}{\partial z} \right) \hat{x} + \left(\frac{\partial(fA_x)}{\partial z} - \frac{\partial(fA_z)}{\partial x} \right) \hat{y} + \left(\frac{\partial(fA_y)}{\partial x} - \frac{\partial(fA_z)}{\partial y} \right) \hat{z} \\ = \left(f \frac{\partial A_x}{\partial y} + A_x \frac{\partial f}{\partial y} - f \frac{\partial A_y}{\partial z} - A_y \frac{\partial f}{\partial z} \right) \hat{x} + \left(f \frac{\partial A_x}{\partial z} + A_x \frac{\partial f}{\partial z} - f \frac{\partial A_z}{\partial x} - A_z \frac{\partial f}{\partial x} \right) \hat{y} \\ + \left(f \frac{\partial A_y}{\partial x} + A_y \frac{\partial f}{\partial x} - f \frac{\partial A_z}{\partial y} - A_z \frac{\partial f}{\partial y} \right) \hat{z} \\ = f \left[\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y} \right) \hat{z} \right] \\ - \left[\left(A_y \frac{\partial f}{\partial z} - A_z \frac{\partial f}{\partial y} \right) \hat{x} + \left(A_z \frac{\partial f}{\partial x} - A_x \frac{\partial f}{\partial z} \right) \hat{y} + \left(A_x \frac{\partial f}{\partial y} - A_y \frac{\partial f}{\partial x} \right) \hat{z} \right] \\ = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f). \quad \text{qed}$$

Problem 1.21

$$(a) (\mathbf{A} \cdot \nabla) \mathbf{B} = \left(A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right) \hat{x} + \left(A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z} \right) \hat{y} \\ + \left(A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z} \right) \hat{z}.$$

$$(b) \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x \hat{x} + y \hat{y} + z \hat{z}}{\sqrt{x^2 + y^2 + z^2}}. \text{ Let's just do the } x \text{ component.}$$

$$[(\hat{\mathbf{r}} \cdot \nabla) \hat{\mathbf{r}}]_x = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\begin{aligned}
 &= \frac{1}{r} \left\{ x \left[\frac{1}{\sqrt{r}} + x \left(-\frac{1}{2} \right) \left(\frac{1}{\sqrt{r}} \right)^2 2x \right] + yx \left[-\frac{1}{2} \left(\frac{1}{\sqrt{r}} \right)^2 2y \right] + zx \left[-\frac{1}{2} \left(\frac{1}{\sqrt{r}} \right)^2 2z \right] \right\} \\
 &= \frac{1}{r} \left\{ \frac{x}{r} - \frac{1}{r^2} (x^3 + xy^2 + xz^2) \right\} = \frac{1}{r} \left\{ \frac{x}{r} - \frac{x}{r^3} (x^2 + y^2 + z^2) \right\} = \frac{1}{r} \left(\frac{x}{r} - \frac{x}{r} \right) = 0.
 \end{aligned}$$

Same goes for the other components. Hence: $\boxed{(\hat{\mathbf{r}} \cdot \nabla) \hat{\mathbf{r}} = 0}$.

$$\begin{aligned}
 \text{(c) } (\mathbf{v}_a \cdot \nabla) \mathbf{v}_b &= \left(x^2 \frac{\partial}{\partial x} + 3xz^2 \frac{\partial}{\partial y} - 2xz \frac{\partial}{\partial z} \right) (xy \hat{\mathbf{x}} + 2yz \hat{\mathbf{y}} + 3xz \hat{\mathbf{z}}) \\
 &= x^2 (y \hat{\mathbf{x}} + 0 \hat{\mathbf{y}} + 3z \hat{\mathbf{z}}) + 3xz^2 (x \hat{\mathbf{x}} + 2z \hat{\mathbf{y}} + 0 \hat{\mathbf{z}}) - 2xz (0 \hat{\mathbf{x}} + 2y \hat{\mathbf{y}} + 3x \hat{\mathbf{z}}) \\
 &= (x^2 y + 3x^2 z^2) \hat{\mathbf{x}} + (6xz^3 - 4xyz) \hat{\mathbf{y}} + (3x^2 z - 6xz^2) \hat{\mathbf{z}} \\
 &= \boxed{x^2 (y + 3z^2) \hat{\mathbf{x}} + 2xz (3z^2 - 2y) \hat{\mathbf{y}} - 3x^2 z \hat{\mathbf{z}}}
 \end{aligned}$$

Problem 1.22

$$\text{(ii) } [\nabla(\mathbf{A} \cdot \mathbf{B})]_x = \frac{\partial}{\partial x} (A_x B_x + A_y B_y + A_z B_z) = \frac{\partial A_x}{\partial x} B_x + A_x \frac{\partial B_x}{\partial x} + \frac{\partial A_y}{\partial x} B_y + A_y \frac{\partial B_y}{\partial x} + \frac{\partial A_z}{\partial x} B_z + A_z \frac{\partial B_z}{\partial x}$$

$$[\mathbf{A} \times (\nabla \times \mathbf{B})]_x = A_y (\nabla \times \mathbf{B})_z - A_z (\nabla \times \mathbf{B})_y = A_y \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) - A_z \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right)$$

$$[\mathbf{B} \times (\nabla \times \mathbf{A})]_x = B_y \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) - B_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)$$

$$[(\mathbf{A} \cdot \nabla) \mathbf{B}]_x = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) B_x = A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}$$

$$[(\mathbf{B} \cdot \nabla) \mathbf{A}]_x = B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z}$$

So $[\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}]_x$

$$\begin{aligned}
 &= A_y \frac{\partial B_x}{\partial x} - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} + A_x \frac{\partial B_x}{\partial x} + B_y \frac{\partial A_x}{\partial x} - B_y \frac{\partial A_x}{\partial y} - B_z \frac{\partial A_x}{\partial z} + B_x \frac{\partial A_x}{\partial x} \\
 &\quad + A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\
 &= B_x \frac{\partial A_x}{\partial x} + A_x \frac{\partial B_x}{\partial x} + B_y \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} \right) + A_y \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} \right) \\
 &\quad + B_z \left(-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial z} \right) + A_z \left(-\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} + \frac{\partial B_x}{\partial z} \right) \\
 &= [\nabla(\mathbf{A} \cdot \mathbf{B})]_x \text{ (same for } y \text{ and } z)
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi) } [\nabla \times (\mathbf{A} \times \mathbf{B})]_x &= \frac{\partial}{\partial y} (\mathbf{A} \times \mathbf{B})_z - \frac{\partial}{\partial z} (\mathbf{A} \times \mathbf{B})_y = \frac{\partial}{\partial y} (A_x B_y - A_y B_x) - \frac{\partial}{\partial z} (A_x B_z - A_z B_x) \\
 &= \frac{\partial A_x}{\partial y} B_y + A_x \frac{\partial B_y}{\partial y} - \frac{\partial A_y}{\partial y} B_x - A_y \frac{\partial B_x}{\partial y} - \frac{\partial A_x}{\partial z} B_z - A_x \frac{\partial B_z}{\partial z} + \frac{\partial A_z}{\partial z} B_x + A_z \frac{\partial B_x}{\partial z}
 \end{aligned}$$

$$[(\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})]_x$$

$$\begin{aligned}
 &= B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} - A_x \frac{\partial B_x}{\partial x} - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} + A_x \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - B_x \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \\
 &= B_y \frac{\partial A_x}{\partial y} + A_x \left(-\frac{\partial B_x}{\partial y} + \frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial y} \right) + B_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - \frac{\partial A_y}{\partial y} - \frac{\partial A_x}{\partial z} \right) \\
 &\quad + A_y \left(-\frac{\partial B_x}{\partial y} \right) + A_z \left(-\frac{\partial B_x}{\partial z} \right) + B_x \left(\frac{\partial A_x}{\partial x} \right) \\
 &= [\nabla \times (\mathbf{A} \times \mathbf{B})]_x \text{ (same for } y \text{ and } z)
 \end{aligned}$$

Problem 1.23

$$\begin{aligned}
 \nabla(f/g) &= \frac{\partial}{\partial x} (f/g) \hat{\mathbf{x}} + \frac{\partial}{\partial y} (f/g) \hat{\mathbf{y}} + \frac{\partial}{\partial z} (f/g) \hat{\mathbf{z}} \\
 &= \frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2} \hat{\mathbf{x}} + \frac{g \frac{\partial f}{\partial y} - f \frac{\partial g}{\partial y}}{g^2} \hat{\mathbf{y}} + \frac{g \frac{\partial f}{\partial z} - f \frac{\partial g}{\partial z}}{g^2} \hat{\mathbf{z}} \\
 &= \frac{1}{g^2} \left[g \left(\frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \right) - f \left(\frac{\partial g}{\partial x} \hat{\mathbf{x}} + \frac{\partial g}{\partial y} \hat{\mathbf{y}} + \frac{\partial g}{\partial z} \hat{\mathbf{z}} \right) \right] = \frac{g \nabla f - f \nabla g}{g^2}. \quad \text{qed}
 \end{aligned}$$

$$\begin{aligned}
 \nabla \cdot (\mathbf{A}/g) &= \frac{\partial}{\partial x} (A_x/g) + \frac{\partial}{\partial y} (A_y/g) + \frac{\partial}{\partial z} (A_z/g) \\
 &= \frac{g \frac{\partial A_x}{\partial x} - A_x \frac{\partial g}{\partial x}}{g^2} + \frac{g \frac{\partial A_y}{\partial y} - A_y \frac{\partial g}{\partial y}}{g^2} + \frac{g \frac{\partial A_z}{\partial z} - A_z \frac{\partial g}{\partial z}}{g^2} \\
 &= \frac{1}{g^2} \left[g \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) - \left(A_x \frac{\partial g}{\partial x} + A_y \frac{\partial g}{\partial y} + A_z \frac{\partial g}{\partial z} \right) \right] = \frac{g \nabla \cdot \mathbf{A} - \mathbf{A} \cdot \nabla g}{g^2}. \quad \text{qed}
 \end{aligned}$$

$$\begin{aligned}
 [\nabla \times (\mathbf{A}/g)]_x &= \frac{\partial}{\partial y} (A_z/g) - \frac{\partial}{\partial z} (A_y/g) \\
 &= \frac{g \frac{\partial A_z}{\partial y} - A_z \frac{\partial g}{\partial y}}{g^2} - \frac{g \frac{\partial A_y}{\partial z} - A_y \frac{\partial g}{\partial z}}{g^2} \\
 &= \frac{1}{g^2} \left[g \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \left(A_z \frac{\partial g}{\partial y} - A_y \frac{\partial g}{\partial z} \right) \right] \\
 &= \frac{g(\nabla \times \mathbf{A})_x + (\mathbf{A} \times \nabla g)_x}{g^2} \quad (\text{same for } y \text{ and } z). \quad \text{qed}
 \end{aligned}$$

Problem 1.24

$$(a) \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} = \hat{x}(6xz) + \hat{y}(9zy) + \hat{z}(-2x^2 - 6y^2)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9zy) + \frac{\partial}{\partial z}(-2x^2 - 6y^2) = 6z + 9z + 0 = 15z$$

$$\nabla \times \mathbf{A} = \hat{x} \left(\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y) \right) + \hat{y} \left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(3z) \right) + \hat{z} \left(\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(x) \right) = 0; \quad \mathbf{B} \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times \mathbf{B} = \hat{x} \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2x) \right) + \hat{y} \left(\frac{\partial}{\partial z}(3y) - \frac{\partial}{\partial x}(0) \right) + \hat{z} \left(\frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(3y) \right) = -5\hat{z}; \quad \mathbf{A} \cdot (\nabla \times \mathbf{B}) = -15z$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \stackrel{?}{=} \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = 0 - (-15z) = 15z. \quad \checkmark$$

$$(b) \mathbf{A} \cdot \mathbf{B} = 3xy - 4xy = -xy; \quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \nabla(-xy) = \hat{x} \frac{\partial}{\partial x}(-xy) + \hat{y} \frac{\partial}{\partial y}(-xy) = -y\hat{x} - x\hat{y}$$

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = \hat{x}(-10y) + \hat{y}(5x); \quad \mathbf{B} \times (\nabla \times \mathbf{A}) = 0$$

$$(\mathbf{A} \cdot \nabla) \mathbf{B} = \left(x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z} \right) (3y\hat{x} - 2x\hat{y}) = \hat{x}(6y) + \hat{y}(-2x)$$

$$(\mathbf{B} \cdot \nabla) \mathbf{A} = \left(3y \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y} \right) (x\hat{x} + 2y\hat{y} + 3z\hat{z}) = \hat{x}(3y) + \hat{y}(-4x)$$

$$\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} \\ = -10y\hat{x} + 5x\hat{y} + 6y\hat{x} - 2x\hat{y} + 3y\hat{x} - 4x\hat{y} = -y\hat{x} - x\hat{y} = \nabla \cdot (\mathbf{A} \cdot \mathbf{B}). \quad \checkmark$$

$$(c) \nabla \times (\mathbf{A} \times \mathbf{B}) = \hat{x} \left(\frac{\partial}{\partial y}(-2x^2 - 6y^2) - \frac{\partial}{\partial z}(9zy) \right) + \hat{y} \left(\frac{\partial}{\partial z}(6xz) - \frac{\partial}{\partial x}(-2x^2 - 6y^2) \right) + \hat{z} \left(\frac{\partial}{\partial x}(9zy) - \frac{\partial}{\partial y}(6xz) \right) \\ = \hat{x}(-12y - 9y) + \hat{y}(6x + 4x) + \hat{z}(0) = -21y\hat{x} + 10x\hat{y}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) = 1 + 2 + 3 = 6; \quad \nabla \cdot \mathbf{B} = \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(-2x) = 0$$

$$(\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) = 3y\hat{x} - 4x\hat{y} - 6y\hat{x} + 2x\hat{y} - 18y\hat{x} + 12x\hat{y} = -21y\hat{x} + 10x\hat{y} \\ = \nabla \times (\mathbf{A} \times \mathbf{B}). \quad \checkmark$$

Problem 1.25

$$(a) \frac{\partial^2 T_a}{\partial x^2} = 2; \quad \frac{\partial^2 T_a}{\partial y^2} = \frac{\partial^2 T_a}{\partial z^2} = 0 \Rightarrow \boxed{\nabla^2 T_a = 2.}$$

$$(b) \frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -T_b \Rightarrow \boxed{\nabla^2 T_b = -3T_b = -3 \sin x \sin y \sin z.}$$

$$(c) \frac{\partial^2 T_c}{\partial x^2} = 25T_c; \quad \frac{\partial^2 T_c}{\partial y^2} = -16T_c; \quad \frac{\partial^2 T_c}{\partial z^2} = -9T_c \Rightarrow \boxed{\nabla^2 T_c = 0.}$$

$$(d) \left. \begin{aligned} \frac{\partial^2 v_x}{\partial x^2} = 2; \quad \frac{\partial^2 v_x}{\partial y^2} = \frac{\partial^2 v_x}{\partial z^2} = 0 &\Rightarrow \nabla^2 v_x = 2 \\ \frac{\partial^2 v_y}{\partial x^2} = \frac{\partial^2 v_y}{\partial y^2} = 0; \quad \frac{\partial^2 v_y}{\partial z^2} = 6x &\Rightarrow \nabla^2 v_y = 6x \\ \frac{\partial^2 v_z}{\partial x^2} = \frac{\partial^2 v_z}{\partial y^2} = \frac{\partial^2 v_z}{\partial z^2} = 0 &\Rightarrow \nabla^2 v_z = 0 \end{aligned} \right\} \boxed{\nabla^2 \mathbf{v} = 2\hat{x} + 6x\hat{y}.}$$

Problem 1.26

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{v}) &= \frac{\partial}{\partial x} \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial y} \right) \\ &= \left(\frac{\partial^2 v_x}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial y \partial x} \right) + \left(\frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial z \partial y} \right) + \left(\frac{\partial^2 v_x}{\partial z \partial z} - \frac{\partial^2 v_z}{\partial z \partial z} \right) = 0, \text{ by equality of cross-derivatives.}\end{aligned}$$

$$\text{From Prob. 1.18: } \nabla \times \mathbf{v}_b = -2y \hat{x} - 3z \hat{y} - x \hat{z} \Rightarrow \nabla \cdot (\nabla \times \mathbf{v}_b) = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(-3z) + \frac{\partial}{\partial z}(-x) = 0. \checkmark$$

Problem 1.27

$$\begin{aligned}\nabla \times (\nabla t) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \end{vmatrix} = \hat{x} \left(\frac{\partial^2 t}{\partial y \partial x} - \frac{\partial^2 t}{\partial x \partial y} \right) + \hat{y} \left(\frac{\partial^2 t}{\partial z \partial x} - \frac{\partial^2 t}{\partial x \partial z} \right) + \hat{z} \left(\frac{\partial^2 t}{\partial z \partial y} - \frac{\partial^2 t}{\partial y \partial z} \right) \\ &= 0, \text{ by equality of cross-derivatives.}\end{aligned}$$

$$\text{In Prob. 1.11(b), } \nabla f = 2xy^3z^4 \hat{x} + 3x^2y^2z^4 \hat{y} + 4x^2y^3z^3 \hat{z}, \text{ so}$$

$$\begin{aligned}\nabla \times (\nabla f) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix} \\ &= \hat{x}(3 \cdot 4x^2y^2z^3 - 4 \cdot 3x^2y^2z^3) + \hat{y}(4 \cdot 2xy^3z^3 - 2 \cdot 4xy^3z^3) + \hat{z}(2 \cdot 3xy^2z^4 - 3 \cdot 2xy^2z^4) = 0. \checkmark\end{aligned}$$

Problem 1.28

$$(a) (0, 0, 0) \rightarrow (1, 0, 0). x: 0 \rightarrow 1, y = z = 0; d\mathbf{l} = dx \hat{x}; \mathbf{v} \cdot d\mathbf{l} = x^2 dx; \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 x^2 dx = (x^3/3)|_0^1 = 1/3.$$

$$(1, 0, 0) \rightarrow (1, 1, 0). x = 1, y: 0 \rightarrow 1, z = 0; d\mathbf{l} = dy \hat{y}; \mathbf{v} \cdot d\mathbf{l} = 2yz dy = 0; \int \mathbf{v} \cdot d\mathbf{l} = 0.$$

$$(1, 1, 0) \rightarrow (1, 1, 1). x = y = 1, z: 0 \rightarrow 1; d\mathbf{l} = dz \hat{z}; \mathbf{v} \cdot d\mathbf{l} = y^2 dz = dz; \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 dz = z|_0^1 = 1.$$

$$\text{Total: } \int \mathbf{v} \cdot d\mathbf{l} = (1/3) + 0 + 1 = \boxed{4/3}.$$

$$(b) (0, 0, 0) \rightarrow (0, 0, 1). x = y = 0, z: 0 \rightarrow 1; d\mathbf{l} = dz \hat{z}; \mathbf{v} \cdot d\mathbf{l} = y^2 dz = 0; \int \mathbf{v} \cdot d\mathbf{l} = 0.$$

$$(0, 0, 1) \rightarrow (0, 1, 1). x = 0, y: 0 \rightarrow 1, z = 1; d\mathbf{l} = dy \hat{y}; \mathbf{v} \cdot d\mathbf{l} = 2yz dy = 2y dy; \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 2y dy = y^2|_0^1 = 1.$$

$$(0, 1, 1) \rightarrow (1, 1, 1). x: 0 \rightarrow 1, y = z = 1; d\mathbf{l} = dx \hat{x}; \mathbf{v} \cdot d\mathbf{l} = x^2 dx; \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 x^2 dx = (x^3/3)|_0^1 = 1/3.$$

$$\text{Total: } \int \mathbf{v} \cdot d\mathbf{l} = 0 + 1 + (1/3) = \boxed{4/3}.$$

$$(c) x = y = z: 0 \rightarrow 1; dx = dy = dz; \mathbf{v} \cdot d\mathbf{l} = x^2 dx + 2yz dy + y^2 dz = x^2 dx + 2x^2 dx + x^2 dx = 4x^2 dx;$$

$$\int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4x^2 dx = (4x^3/3)|_0^1 = \boxed{4/3}.$$

$$(d) \oint \mathbf{v} \cdot d\mathbf{l} = (4/3) - (4/3) = \boxed{0}.$$

Problem 1.29

$x, y: 0 \rightarrow 1, z = 0; d\mathbf{a} = dx dy \hat{z}; \mathbf{v} \cdot d\mathbf{a} = y(z^2 - 3) dx dy = -3y dx dy; \int \mathbf{v} \cdot d\mathbf{a} = -3 \int_0^1 dx \int_0^1 y dy = -3(x|_0^1)(y^2/2|_0^1) = -3(2)(2) = \boxed{12}$. In Ex. 1.7 we got 20, for the same boundary line (the square in the xy -plane), so the answer is **no**: the surface integral does not depend only on the boundary line. The total flux for the cube is $20 + 12 = \boxed{32}$.

Problem 1.30

$\int T dr = \int z^2 dx dy dz$. You can do the integrals in any order—here it is simplest to save z for last:

$$\int z^2 \left[\int \left(\int dx \right) dy \right] dz.$$

The sloping surface is $x + y + z = 1$, so the x integral is $\int_0^{1-y-z} dx = 1 - y - z$. For a given x, y ranges from 0 to $1 - z$, so the y integral is $\int_0^{1-z} (1 - y - z) dy = [(1 - z)y - (y^2/2)]|_0^{1-z} = (1 - z)^2 - [(1 - z)^2/2] = (1 - z)^2/2 =$

$(1/2) - z + (z^2/2)$. Finally, the z integral is $\int_0^1 z^2(\frac{1}{2} - z + \frac{z^2}{2}) dz = \int_0^1 (\frac{z^2}{2} - z^3 + \frac{z^4}{2}) dz = (\frac{z^3}{6} - \frac{z^4}{4} + \frac{z^5}{10}) \Big|_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60}$.

Problem 1.31

$$T(\mathbf{b}) = 1 + 4 + 2 = 7; \quad T(\mathbf{a}) = 0. \quad \Rightarrow \quad T(\mathbf{b}) - T(\mathbf{a}) = 7.$$

$$\nabla T = (2x + 4y)\hat{x} + (4x + 2z^3)\hat{y} + (6yz^2)\hat{z}; \quad \nabla T \cdot d\mathbf{l} = (2x + 4y)dx + (4x + 2z^3)dy + (6yz^2)dz$$

- (a) Segment 1: $x: 0 \rightarrow 1, y = z = dy = dz = 0. \int \nabla T \cdot d\mathbf{l} = \int_0^1 (2x) dx = x^2 \Big|_0^1 = 1.$
 Segment 2: $y: 0 \rightarrow 1, x = 1, z = 0, dx = dz = 0. \int \nabla T \cdot d\mathbf{l} = \int_0^1 (4) dy = 4y \Big|_0^1 = 4.$
 Segment 3: $z: 0 \rightarrow 1, x = y = 1, dx = dy = 0. \int \nabla T \cdot d\mathbf{l} = \int_0^1 (6z^2) dz = 2z^3 \Big|_0^1 = 2.$ } $\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l} = 7. \checkmark$
- (b) Segment 1: $z: 0 \rightarrow 1, x = y = dx = dy = 0. \int \nabla T \cdot d\mathbf{l} = \int_0^1 (0) dz = 0.$
 Segment 2: $y: 0 \rightarrow 1, x = 0, z = 1, dx = dz = 0. \int \nabla T \cdot d\mathbf{l} = \int_0^1 (2) dy = 2y \Big|_0^1 = 2.$
 Segment 3: $x: 0 \rightarrow 1, y = z = 1, dy = dz = 0. \int \nabla T \cdot d\mathbf{l} = \int_0^1 (2x + 4) dx = (x^2 + 4x) \Big|_0^1 = 1 + 4 = 5.$ } $\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l} = 7. \checkmark$
- (c) $x: 0 \rightarrow 1, y = x, z = x^2, dy = dx, dz = 2x dx.$

$$\nabla T \cdot d\mathbf{l} = (2x + 4x)dx + (4x + 2x^6)dx + (6xx^4)2x dx = (10x + 14x^6)dx.$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l} = \int_0^1 (10x + 14x^6)dx = (5x^2 + 2x^7) \Big|_0^1 = 5 + 2 = 7. \checkmark$$

Problem 1.32

$$\nabla \cdot \mathbf{v} = y + 2z + 3x$$

$$\begin{aligned} \int (\nabla \cdot \mathbf{v}) d\tau &= \int (y + 2z + 3x) dx dy dz = \iint \left\{ \int_0^2 (y + 2z + 3x) dx \right\} dy dz \\ &\quad \hookrightarrow [(y + 2z)x + \frac{3}{2}x^2]_0^2 = 2(y + 2z) + 6 \\ &= \int \left\{ \int_0^2 (2y + 4z + 6) dy \right\} dz \\ &\quad \hookrightarrow [y^2 + (4z + 6)y]_0^2 = 4 + 2(4z + 6) = 8z + 16 \\ &= \int_0^2 (8z + 16) dz = (4z^2 + 16z) \Big|_0^2 = 16 + 32 = \boxed{48}. \end{aligned}$$

Numbering the surfaces as in Fig. 1.29:

- (i) $d\mathbf{a} = dy dz \hat{x}, x = 2. \mathbf{v} \cdot d\mathbf{a} = 2y dy dz. \int \mathbf{v} \cdot d\mathbf{a} = \iint 2y dy dz = 2y^2 \Big|_0^2 = 8.$
 (ii) $d\mathbf{a} = -dy dz \hat{x}, x = 0. \mathbf{v} \cdot d\mathbf{a} = 0. \int \mathbf{v} \cdot d\mathbf{a} = 0.$
 (iii) $d\mathbf{a} = dx dz \hat{y}, y = 2. \mathbf{v} \cdot d\mathbf{a} = 4z dx dz. \int \mathbf{v} \cdot d\mathbf{a} = \iint 4z dx dz = 16.$
 (iv) $d\mathbf{a} = -dx dz \hat{y}, y = 0. \mathbf{v} \cdot d\mathbf{a} = 0. \int \mathbf{v} \cdot d\mathbf{a} = 0.$
 (v) $d\mathbf{a} = dx dy \hat{z}, z = 2. \mathbf{v} \cdot d\mathbf{a} = 6x dx dy. \int \mathbf{v} \cdot d\mathbf{a} = 24.$
 (vi) $d\mathbf{a} = -dx dy \hat{z}, z = 0. \mathbf{v} \cdot d\mathbf{a} = 0. \int \mathbf{v} \cdot d\mathbf{a} = 0.$
 $\Rightarrow \int \mathbf{v} \cdot d\mathbf{a} = 8 + 16 + 24 = 48 \checkmark$
-

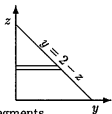
Problem 1.33

$$\nabla \times \mathbf{v} = \hat{x}(0 - 2y) + \hat{y}(0 - 3z) + \hat{z}(0 - x) = -2y\hat{x} - 3z\hat{y} - x\hat{z}.$$

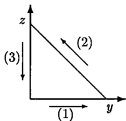
$d\mathbf{a} = dy dz \hat{x}$, if we agree that the path integral shall run counterclockwise. So

$$(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = -2y dy dz.$$

$$\begin{aligned}
 f(\nabla \times \mathbf{v}) \cdot d\mathbf{a} &= \int \left\{ \int_0^{2-z} (-2y) dy \right\} dz \\
 &\quad \hookrightarrow y^2 \Big|_0^{2-z} = -(2-z)^2 \\
 &= -\int_0^2 (4-4z+z^2) dz = -\left(4z-2z^2+\frac{z^3}{3}\right) \Big|_0^2 \\
 &= -(8-8+\frac{8}{3}) = \boxed{-\frac{8}{3}}
 \end{aligned}$$



Meanwhile, $\mathbf{v} \cdot d\mathbf{l} = (xy)dx + (2yz)dy + (3xz)dz$. There are three segments.



- (1) $x = z = 0$; $dx = dz = 0$, $y : 0 \rightarrow 2$. $\int \mathbf{v} \cdot d\mathbf{l} = 0$.
 (2) $x = 0$; $z = 2 - y$; $dx = 0$, $dz = -dy$, $y : 2 \rightarrow 0$. $\mathbf{v} \cdot d\mathbf{l} = 2yz dy$.
 $\int \mathbf{v} \cdot d\mathbf{l} = \int_2^0 2y(2-y) dy = -\int_0^2 (4y-2y^2) dy = -\left(2y^2 - \frac{2}{3}y^3\right) \Big|_0^2 = -(8 - \frac{2}{3} \cdot 8) = -\frac{8}{3}$.
 (3) $x = y = 0$; $dx = dy = 0$; $z : 2 \rightarrow 0$. $\mathbf{v} \cdot d\mathbf{l} = 0$. $\int \mathbf{v} \cdot d\mathbf{l} = 0$. So $\oint \mathbf{v} \cdot d\mathbf{l} = -\frac{8}{3}$. ✓

Problem 1.34

By Corollary 1, $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ should equal $\frac{4}{3}$. $\nabla \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}$.

- (i) $d\mathbf{a} = dy dz \hat{\mathbf{x}}$, $x = 1$; $y, z : 0 \rightarrow 1$. $(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = (4z^2 - 2)dy dz$; $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 (4z^2 - 2) dz$
 $= \left(\frac{4}{3}z^3 - 2z\right) \Big|_0^1 = \frac{4}{3} - 2 = -\frac{2}{3}$.
 (ii) $d\mathbf{a} = -dx dy \hat{\mathbf{z}}$, $z = 0$; $x, y : 0 \rightarrow 1$. $(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$; $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$.
 (iii) $d\mathbf{a} = dx dz \hat{\mathbf{y}}$, $y = 1$; $x, z : 0 \rightarrow 1$. $(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$; $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$.
 (iv) $d\mathbf{a} = -dx dz \hat{\mathbf{y}}$, $y = 0$; $x, z : 0 \rightarrow 1$. $(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$; $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$.
 (v) $d\mathbf{a} = dx dy \hat{\mathbf{z}}$, $z = 1$; $x, y : 0 \rightarrow 1$. $(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 2 dx dy$; $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 2$.
 $\Rightarrow \int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = -\frac{2}{3} + 2 = \frac{4}{3}$. ✓

Problem 1.35

- (a) Use the product rule $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$:

$$\int_S f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_S \nabla \times (f\mathbf{A}) \cdot d\mathbf{a} + \int_S [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} = \oint_P f\mathbf{A} \cdot d\mathbf{l} + \int_S [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a}. \quad \text{qed.}$$

(I used Stokes' theorem in the last step.)

- (b) Use the product rule $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$:

$$\int_V \mathbf{B} \cdot (\nabla \times \mathbf{A}) d\tau = \int_V \nabla \cdot (\mathbf{A} \times \mathbf{B}) d\tau + \int_V \mathbf{A} \cdot (\nabla \times \mathbf{B}) d\tau = \oint_S (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} + \int_V \mathbf{A} \cdot (\nabla \times \mathbf{B}) d\tau. \quad \text{qed.}$$

(I used the divergence theorem in the last step.)

$$\text{Problem 1.36} \quad r = \sqrt{x^2 + y^2 + z^2}; \quad \theta = \cos^{-1} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right); \quad \phi = \tan^{-1} \left(\frac{y}{x} \right).$$

Problem 1.37

There are many ways to do this one—probably the most illuminating way is to work it out by trigonometry from Fig. 1.36. The most systematic approach is to study the expression:

$$r = x \hat{x} + y \hat{y} + z \hat{z} = r \sin \theta \cos \phi \hat{x} + r \sin \theta \sin \phi \hat{y} + r \cos \theta \hat{z}.$$

If I only vary r slightly, then $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial r}(r)dr$ is a short vector pointing in the direction of increase in r . To make it a unit vector, I must divide by its length. Thus:

$$\hat{\mathbf{r}} = \frac{\frac{\partial \mathbf{r}}{\partial r}}{\left| \frac{\partial \mathbf{r}}{\partial r} \right|}; \quad \hat{\theta} = \frac{\frac{\partial \mathbf{r}}{\partial \theta}}{\left| \frac{\partial \mathbf{r}}{\partial \theta} \right|}; \quad \hat{\phi} = \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left| \frac{\partial \mathbf{r}}{\partial \phi} \right|}.$$

$$\frac{\partial \mathbf{r}}{\partial r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}; \quad \left| \frac{\partial \mathbf{r}}{\partial r} \right|^2 = \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta = 1.$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = r \cos \theta \cos \phi \hat{x} + r \cos \theta \sin \phi \hat{y} - r \sin \theta \hat{z}; \quad \left| \frac{\partial \mathbf{r}}{\partial \theta} \right|^2 = r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta = r^2.$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = -r \sin \theta \sin \phi \hat{x} + r \sin \theta \cos \phi \hat{y}; \quad \left| \frac{\partial \mathbf{r}}{\partial \phi} \right|^2 = r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi = r^2 \sin^2 \theta.$$

$$\begin{aligned} \hat{\mathbf{r}} &= \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}. \\ \hat{\theta} &= \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}. \\ \hat{\phi} &= -\sin \phi \hat{x} + \cos \phi \hat{y}. \end{aligned}$$

$$\text{Check: } \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta = \sin^2 \theta + \cos^2 \theta = 1, \quad \checkmark$$

$$\hat{\theta} \cdot \hat{\theta} = -\cos \theta \sin \phi \cos \phi + \cos \theta \sin \phi \cos \phi = 0, \quad \checkmark \text{ etc.}$$

$$\sin \theta \hat{\mathbf{r}} = \sin^2 \theta \cos \phi \hat{x} + \sin^2 \theta \sin \phi \hat{y} + \sin \theta \cos \theta \hat{z}.$$

$$\cos \theta \hat{\theta} = \cos^2 \theta \cos \phi \hat{x} + \cos^2 \theta \sin \phi \hat{y} - \sin \theta \cos \theta \hat{z}.$$

Add these:

$$(1) \quad \sin \theta \hat{\mathbf{r}} + \cos \theta \hat{\theta} = +\cos \phi \hat{x} + \sin \phi \hat{y};$$

$$(2) \quad \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}.$$

Multiply (1) by $\cos \phi$, (2) by $\sin \phi$, and subtract:

$$\hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}.$$

Multiply (1) by $\sin \phi$, (2) by $\cos \phi$, and add:

$$\hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}.$$

$$\cos \theta \hat{\mathbf{r}} = \sin \theta \cos \theta \cos \phi \hat{x} + \sin \theta \cos \theta \sin \phi \hat{y} + \cos^2 \theta \hat{z}.$$

$$\sin \theta \hat{\theta} = \sin \theta \cos \theta \cos \phi \hat{x} + \sin \theta \cos \theta \sin \phi \hat{y} - \sin^2 \theta \hat{z}.$$

Subtract these:

$$\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta}.$$

Problem 1.38

(a) $\nabla \cdot \mathbf{v}_1 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2) = \frac{1}{r^2} 4r^3 = 4r$

$$\int (\nabla \cdot \mathbf{v}_1) dr = \int (4r) (r^2 \sin \theta dr d\theta d\phi) = (4) \int_0^R r^3 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = (4) \left(\frac{R^4}{4}\right) (2)(2\pi) = \boxed{4\pi R^4}$$

$$\int \mathbf{v}_1 \cdot d\mathbf{a} = \int (r^2 \hat{\mathbf{r}}) \cdot (r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}) = r^4 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = 4\pi R^4 \checkmark \text{ (Note: at surface of sphere } r = R.)$$

(b) $\nabla \cdot \mathbf{v}_2 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{1}{r^2}) = 0 \Rightarrow \boxed{\int (\nabla \cdot \mathbf{v}_2) dr = 0}$

$$\int \mathbf{v}_2 \cdot d\mathbf{a} = \int (\frac{1}{r^2} \hat{\mathbf{r}}) (r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}) = \int \sin \theta d\theta d\phi = \boxed{4\pi}$$

They *don't* agree! The point is that this divergence is zero *except at the origin*, where it blows up, so our calculation of $\int (\nabla \cdot \mathbf{v}_2)$ is *incorrect*. The right answer is 4π .

Problem 1.39

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta r \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (r \sin \theta \cos \phi) \\ &= \frac{1}{r^2} 3r^2 \cos \theta + \frac{1}{r \sin \theta} r 2 \sin \theta \cos \theta + \frac{1}{r \sin \theta} r \sin \theta (-\sin \phi) \\ &= 3 \cos \theta + 2 \cos \theta - \sin \phi = 5 \cos \theta - \sin \phi \end{aligned}$$

$$\int (\nabla \cdot \mathbf{v}) dr = \int (5 \cos \theta - \sin \phi) r^2 \sin \theta dr d\theta d\phi = \int_0^R r^2 dr \int_0^\pi [\int_0^{2\pi} (5 \cos \theta - \sin \phi) d\phi] d\theta \sin \theta$$

$\hookrightarrow 2\pi(5 \cos \theta)$

$$= \left(\frac{R^3}{3}\right) (10\pi) \int_0^\pi \sin \theta \cos \theta d\theta$$

$\hookrightarrow \frac{\sin^2 \theta}{2} \Big|_0^\pi = \frac{1}{2}$

$$= \boxed{\frac{5\pi}{3} R^3}$$

Two surfaces—one the hemisphere: $d\mathbf{a} = R^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}$; $r = R$; $\phi: 0 \rightarrow 2\pi$, $\theta: 0 \rightarrow \frac{\pi}{2}$.

$$\int \mathbf{v} \cdot d\mathbf{a} = \int (r \cos \theta) R^2 \sin \theta d\theta d\phi = R^3 \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi = R^3 \left(\frac{1}{2}\right) (2\pi) = \pi R^3.$$

other the flat bottom: $d\mathbf{a} = (dr)(r \sin \theta d\phi)(+\hat{\theta}) = r dr d\phi \hat{\theta}$ (here $\theta = \frac{\pi}{2}$). $r: 0 \rightarrow R$, $\phi: 0 \rightarrow 2\pi$.

$$\int \mathbf{v} \cdot d\mathbf{a} = \int (r \sin \theta) (r dr d\phi) = \int_0^R r^2 dr \int_0^{2\pi} d\phi = 2\pi \frac{R^3}{3}.$$

Total: $\int \mathbf{v} \cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3$. \checkmark

Problem 1.40 $\nabla t = (\cos \theta + \sin \theta \cos \phi) \hat{\mathbf{r}} + (-\sin \theta + \cos \theta \cos \phi) \hat{\theta} + \frac{1}{\sin \theta} (-\sin \phi \sin \theta) \hat{\phi}$

$$\begin{aligned} \nabla^2 t &= \nabla \cdot (\nabla t) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 (\cos \theta + \sin \theta \cos \phi)) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta (-\sin \theta + \cos \theta \cos \phi)) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (-\sin \phi) \\ &= \frac{1}{r^2} 2r (\cos \theta + \sin \theta \cos \phi) + \frac{1}{r \sin \theta} (-2 \sin \theta \cos \theta + \cos^2 \theta \cos \phi - \sin^2 \theta \cos \phi) - \frac{1}{r \sin \theta} \cos \phi \\ &= \frac{1}{r \sin \theta} [2 \sin \theta \cos \theta + 2 \sin^2 \theta \cos \phi - 2 \sin \theta \cos \theta + \cos^2 \theta \cos \phi - \sin^2 \theta \cos \phi - \cos \phi] \\ &= \frac{1}{r \sin \theta} [(\sin^2 \theta + \cos^2 \theta) \cos \phi - \cos \phi] = 0. \\ &\Rightarrow \boxed{\nabla^2 t = 0} \end{aligned}$$

Check: $r \cos \theta = z$, $r \sin \theta \cos \phi = x \Rightarrow$ in Cartesian coordinates $t = x + z$. Obviously, Laplacian is zero.

Gradient Theorem: $\int_a^b \nabla t \cdot d\mathbf{l} = t(\mathbf{b}) - t(\mathbf{a})$

Segment 1: $\theta = \frac{\pi}{2}$, $\phi = 0$, $r: 0 \rightarrow 2$. $d\mathbf{l} = dr \hat{\mathbf{r}}$; $\nabla t \cdot d\mathbf{l} = (\cos \theta + \sin \theta \cos \phi) dr = (0 + 1) dr = dr$.

$$\int \nabla t \cdot d\mathbf{l} = \int_0^2 dr = 2.$$

Segment 2: $\theta = \frac{\pi}{2}$, $r = 2$, $\phi: 0 \rightarrow \frac{\pi}{2}$. $d\mathbf{l} = r \sin \theta d\phi \hat{\phi} = 2 d\phi \hat{\phi}$.

$$\nabla t \cdot d\mathbf{l} = (-\sin \phi) (2 d\phi) = -2 \sin \phi d\phi. \quad \int \nabla t \cdot d\mathbf{l} = -\int_0^{\pi/2} 2 \sin \phi d\phi = 2 \cos \phi \Big|_0^{\pi/2} = -2.$$

Segment 3: $r = 2$, $\phi = \frac{\pi}{2}$; $\theta : \frac{\pi}{2} \rightarrow 0$.

$$\begin{aligned} d\mathbf{l} &= r d\theta \hat{\theta} = 2 d\theta \hat{\theta}; \quad \nabla t \cdot d\mathbf{l} = (-\sin \theta + \cos \theta \cos \phi)(2 d\theta) = -2 \sin \theta d\theta. \\ \int \nabla t \cdot d\mathbf{l} &= -\int_{\frac{\pi}{2}}^0 2 \sin \theta d\theta = 2 \cos \theta \Big|_{\frac{\pi}{2}}^0 = 2. \end{aligned}$$

Total: $\int_a^b \nabla t \cdot d\mathbf{l} = 2 - 2 + 2 = \boxed{2}$. Meanwhile, $t(\mathbf{b}) - t(\mathbf{a}) = [2(1+0)] - [0(\quad)] = 2$. ✓

Problem 1.41 From Fig. 1.42, $\hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}$; $\hat{\phi} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$; $\hat{\mathbf{z}} = \hat{\mathbf{z}}$

Multiply first by $\cos \phi$, second by $\sin \phi$, and subtract:

$$\hat{\mathbf{s}} \cos \phi - \hat{\phi} \sin \phi = \cos^2 \phi \hat{\mathbf{x}} + \cos \phi \sin \phi \hat{\mathbf{y}} + \sin^2 \phi \hat{\mathbf{x}} - \sin \phi \cos \phi \hat{\mathbf{y}} = \hat{\mathbf{x}}(\sin^2 \phi + \cos^2 \phi) = \hat{\mathbf{x}}.$$

So $\hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\phi}$.

Multiply first by $\sin \phi$, second by $\cos \phi$, and add:

$$\hat{\mathbf{s}} \sin \phi + \hat{\phi} \cos \phi = \sin \phi \cos \phi \hat{\mathbf{x}} + \sin^2 \phi \hat{\mathbf{y}} - \sin \phi \cos \phi \hat{\mathbf{x}} + \cos^2 \phi \hat{\mathbf{y}} = \hat{\mathbf{y}}(\sin^2 \phi + \cos^2 \phi) = \hat{\mathbf{y}}.$$

So $\hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\phi}$. $\hat{\mathbf{z}} = \hat{\mathbf{z}}$.

Problem 1.42

$$\begin{aligned} \text{(a)} \quad \nabla \cdot \mathbf{v} &= \frac{1}{s} \frac{\partial}{\partial s} (s s (2 + \sin^2 \phi)) + \frac{1}{s} \frac{\partial}{\partial \phi} (s \sin \phi \cos \phi) + \frac{\partial}{\partial z} (3z) \\ &= \frac{1}{s} 2s(2 + \sin^2 \phi) + \frac{1}{s} s(\cos^2 \phi - \sin^2 \phi) + 3 \\ &= 4 + 2 \sin^2 \phi + \cos^2 \phi - \sin^2 \phi + 3 \\ &= 4 + \sin^2 \phi + \cos^2 \phi + 3 = \boxed{8}. \end{aligned}$$

$$\text{(b)} \quad \int (\nabla \cdot \mathbf{v}) d\tau = \int (8) s ds d\phi dz = 8 \int_0^2 s ds \int_0^{\frac{\pi}{2}} d\phi \int_0^5 dz = 8(2) \left(\frac{\pi}{2}\right) (5) = \boxed{40\pi}.$$

Meanwhile, the surface integral has five parts:

$$\text{top: } z = 5, \quad d\mathbf{a} = s ds d\phi \hat{\mathbf{z}}; \quad \mathbf{v} \cdot d\mathbf{a} = 3z s ds d\phi = 15s ds d\phi. \quad \int \mathbf{v} \cdot d\mathbf{a} = 15 \int_0^2 s ds \int_0^{\frac{\pi}{2}} d\phi = 15\pi.$$

$$\text{bottom: } z = 0, \quad d\mathbf{a} = -s ds d\phi \hat{\mathbf{z}}; \quad \mathbf{v} \cdot d\mathbf{a} = -3z s ds d\phi = 0. \quad \int \mathbf{v} \cdot d\mathbf{a} = 0.$$

$$\text{back: } \phi = \frac{\pi}{2}, \quad d\mathbf{a} = ds dz \hat{\phi}; \quad \mathbf{v} \cdot d\mathbf{a} = s \sin \phi \cos \phi ds dz = 0. \quad \int \mathbf{v} \cdot d\mathbf{a} = 0.$$

$$\text{left: } \phi = 0, \quad d\mathbf{a} = -ds dz \hat{\phi}; \quad \mathbf{v} \cdot d\mathbf{a} = -s \sin \phi \cos \phi ds dz = 0. \quad \int \mathbf{v} \cdot d\mathbf{a} = 0.$$

$$\text{front: } s = 2, \quad d\mathbf{a} = s d\phi dz \hat{\mathbf{s}}; \quad \mathbf{v} \cdot d\mathbf{a} = s(2 + \sin^2 \phi) s d\phi dz = 4(2 + \sin^2 \phi) d\phi dz.$$

$$\int \mathbf{v} \cdot d\mathbf{a} = 4 \int_0^{\frac{\pi}{2}} (2 + \sin^2 \phi) d\phi \int_0^5 dz = (4)(\pi + \frac{\pi}{4})(5) = 25\pi.$$

So $\oint \mathbf{v} \cdot d\mathbf{a} = 15\pi + 25\pi = 40\pi$. ✓

$$\begin{aligned} \text{(c)} \quad \nabla \times \mathbf{v} &= \left(\frac{1}{s} \frac{\partial}{\partial \phi} (3z) - \frac{\partial}{\partial z} (s \sin \phi \cos \phi) \right) \hat{\mathbf{s}} + \left(\frac{\partial}{\partial z} (s(2 + \sin^2 \phi)) - \frac{\partial}{\partial s} (3z) \right) \hat{\phi} \\ &\quad + \frac{1}{s} \left(\frac{\partial}{\partial s} (s^2 \sin \phi \cos \phi) - \frac{\partial}{\partial \phi} (s(2 + \sin^2 \phi)) \right) \hat{\mathbf{z}} \\ &= \frac{1}{s} (2s \sin \phi \cos \phi - s 2 \sin \phi \cos \phi) = \boxed{0}. \end{aligned}$$

Problem 1.43

$$\text{(a)} \quad 3(3^2) - 2(3) - 1 = 27 - 6 - 1 = \boxed{20}.$$

$$\text{(b)} \quad \cos \pi = \boxed{-1}.$$

$$\text{(c)} \quad \boxed{\text{zero}}.$$

$$\text{(d)} \quad \ln(-2 + 3) = \ln 1 = \boxed{\text{zero}}.$$

Problem 1.44

$$\text{(a)} \quad \int_{-2}^2 (2x + 3) \frac{1}{3} \delta(x) dx = \frac{1}{3}(0 + 3) = \boxed{1}.$$

$$\text{(b)} \quad \text{By Eq. 1.94, } \delta(1 - x) = \delta(x - 1), \text{ so } 1 + 3 + 2 = \boxed{6}.$$

$$(c) \int_{-1}^1 9x^2 \frac{1}{3} \delta(x + \frac{1}{3}) dx = 9 \left(-\frac{1}{3}\right)^2 \frac{1}{3} = \boxed{\frac{1}{3}}$$

$$(d) \boxed{1 \text{ (if } a > b), 0 \text{ (if } a < b)}.$$

Problem 1.45

$$(a) \int_{-\infty}^{\infty} f(x) \left[x \frac{d}{dx} \delta(x) \right] dx = x f(x) \delta(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} (x f(x)) \delta(x) dx.$$

The first term is zero, since $\delta(x) = 0$ at $\pm\infty$; $\frac{d}{dx} (x f(x)) = x \frac{df}{dx} + \frac{dx}{dx} f = x \frac{df}{dx} + f$.

So the integral is $-\int_{-\infty}^{\infty} \left(x \frac{df}{dx} + f \right) \delta(x) dx = 0 - f(0) = -f(0) = -\int_{-\infty}^{\infty} f(x) \delta(x) dx$.

So, $x \frac{d}{dx} \delta(x) = -\delta(x)$. qed

$$(b) \int_{-\infty}^{\infty} f(x) \frac{d\theta}{dx} dx = f(x) \theta(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df}{dx} \theta(x) dx = f(\infty) - \int_0^{\infty} \frac{df}{dx} dx = f(\infty) - (f(\infty) - f(0)) \\ = f(0) = \int_{-\infty}^{\infty} f(x) \delta(x) dx. \text{ So } \frac{d\theta}{dx} = \delta(x). \text{ qed}$$

Problem 1.46

$$(a) \boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}')}. \text{ Check: } \int \rho(\mathbf{r}) d\tau = q \int \delta^3(\mathbf{r} - \mathbf{r}') d\tau = q. \quad \checkmark$$

$$(b) \boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}') - q\delta^3(\mathbf{r})}.$$

(c) Evidently $\rho(\mathbf{r}) = A\delta(\mathbf{r} - R)$. To determine the constant A , we require

$$Q = \int \rho d\tau = \int A\delta(\mathbf{r} - R) 4\pi r^2 d\mathbf{r} = A 4\pi R^2. \quad \text{So } A = \frac{Q}{4\pi R^2}. \quad \boxed{\rho(\mathbf{r}) = \frac{Q}{4\pi R^2} \delta(\mathbf{r} - R)}.$$

Problem 1.47

$$(a) a^2 + \mathbf{a} \cdot \mathbf{a} + a^2 = \boxed{3a^2}.$$

$$(b) \int (\mathbf{r} - \mathbf{b})^2 \frac{1}{5^3} \delta^3(\mathbf{r}) d\mathbf{r} = \frac{1}{125} b^2 = \frac{1}{125} (4^2 + 3^2) = \boxed{\frac{1}{5}}.$$

(c) $c^2 = 25 + 9 + 4 = 38 > 36 = 6^2$, so \mathbf{c} is outside \mathcal{V} , so the integral is **zero**.

(d) $(\mathbf{e} - (2\hat{x} + 2\hat{y} + 2\hat{z}))^2 = (1\hat{x} + 0\hat{y} + (-1)\hat{z})^2 = 1 + 1 = 2 < (1.5)^2 = 2.25$, so \mathbf{e} is inside \mathcal{V} , and hence the integral is $\mathbf{e} \cdot (\mathbf{d} - \mathbf{e}) = (3, 2, 1) \cdot (-2, 0, 2) = -6 + 0 + 2 = \boxed{-4}$.

Problem 1.48

First method: use Eq. 1.99 to write $J = \int e^{-r} (4\pi\delta^3(\mathbf{r})) d\mathbf{r} = 4\pi e^{-0} = \boxed{4\pi}$.

Second method: integrating by parts (use Eq. 1.59).

$$J = - \int_{\mathcal{V}} \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla(e^{-r}) d\tau + \oint_S e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a}. \quad \text{But } \nabla(e^{-r}) = \left(\frac{\partial}{\partial r} e^{-r} \right) \hat{\mathbf{r}} = -e^{-r} \hat{\mathbf{r}}. \\ = \int \frac{1}{r^2} e^{-r} 4\pi r^2 d\mathbf{r} + \int e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \cdot r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}} = 4\pi \int_0^{\infty} e^{-r} dr + e^{-R} \int \sin\theta d\theta d\phi \\ = 4\pi (-e^{-r}) \Big|_0^{\infty} + 4\pi e^{-R} = 4\pi (-e^{-\infty} + e^{-0}) = 4\pi \checkmark \quad (\text{Here } R = \infty, \text{ so } e^{-R} = 0.)$$

Problem 1.49 (a) $\nabla \cdot \mathbf{F}_1 = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x^2) = \boxed{0}$; $\nabla \cdot \mathbf{F}_2 = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = \boxed{3}$

$$\nabla \times \mathbf{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = -\hat{y} \frac{\partial}{\partial x}(x^2) = \boxed{-2x\hat{y}}; \quad \nabla \times \mathbf{F}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \boxed{0}$$

\mathbf{F}_2 is a gradient; \mathbf{F}_1 is a curl $U_2 = \frac{1}{2}(x^2 + y^2 + z^2)$ would do ($\mathbf{F}_2 = \nabla U_2$).

For \mathbf{A}_1 , we want $(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) = (\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}) = 0$; $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$. $A_y = \frac{x^3}{3}$, $A_x = A_z = 0$ would do it.

$\mathbf{A}_1 = \frac{1}{3}x^2 \hat{y}$ ($\mathbf{F}_1 = \nabla \times \mathbf{A}_1$). (But these are not unique.)

(b) $\nabla \cdot \mathbf{F}_3 = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$; $\nabla \times \mathbf{F}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{x}(x-x) + \hat{y}(y-y) + \hat{z}(z-z) = 0$

So \mathbf{F}_3 can be written as the gradient of a scalar ($\mathbf{F}_3 = \nabla U_3$) and as the curl of a vector ($\mathbf{F}_3 = \nabla \times \mathbf{A}_3$). In fact, $U_3 = xyz$ does the job. For the vector potential, we have

$$\left\{ \begin{array}{ll} \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = yz, & \text{which suggests } A_x = \frac{1}{4}y^2z + f(x, z); \quad A_y = -\frac{1}{4}yz^2 + g(x, y) \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz, & \text{suggesting } A_x = \frac{1}{4}z^2x + h(x, y); \quad A_z = -\frac{1}{4}zx^2 + j(y, z) \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y} = xy, & \text{so } A_y = \frac{1}{4}x^2y + k(y, z); \quad A_z = -\frac{1}{4}xy^2 + l(x, y) \end{array} \right\}$$

Putting this all together: $\mathbf{A}_3 = \frac{1}{4}\{x(z^2 - y^2)\hat{x} + y(x^2 - z^2)\hat{y} + z(y^2 - x^2)\hat{z}\}$ (again, not unique).

Problem 1.50

(d) \Rightarrow (a): $\nabla \times \mathbf{F} = \nabla \times (-\nabla U) = 0$ (Eq. 1.44 – curl of gradient is always zero).

(a) \Rightarrow (c): $\oint \mathbf{F} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = 0$ (Eq. 1.57–Stokes' theorem).

(c) \Rightarrow (b): $\int_a^b \mathbf{F} \cdot d\mathbf{l} - \int_a^b \mathbf{F} \cdot d\mathbf{l} = \int_a^b \mathbf{F} \cdot d\mathbf{l} + \int_b^a \mathbf{F} \cdot d\mathbf{l} = \oint \mathbf{F} \cdot d\mathbf{l} = 0$, so

$$\int_a^b \mathbf{F} \cdot d\mathbf{l} = \int_a^b \mathbf{F} \cdot d\mathbf{l}.$$

(b) \Rightarrow (c): same as (c) \Rightarrow (b), only in reverse; (c) \Rightarrow (a): same as (a) \Rightarrow (c).

Problem 1.51

(d) \Rightarrow (a): $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{W}) = 0$ (Eq. 1.46—divergence of curl is always zero).

(a) \Rightarrow (c): $\oint \mathbf{F} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{F}) d\tau = 0$ (Eq. 1.56—divergence theorem).

(c) \Rightarrow (b): $\int_I \mathbf{F} \cdot d\mathbf{a} - \int_{II} \mathbf{F} \cdot d\mathbf{a} = \oint \mathbf{F} \cdot d\mathbf{a} = 0$, so

$$\int_I \mathbf{F} \cdot d\mathbf{a} = \int_{II} \mathbf{F} \cdot d\mathbf{a}.$$

(Note: sign change because for $\oint \mathbf{F} \cdot d\mathbf{a}$, $d\mathbf{a}$ is *outward*, whereas for surface II it is *inward*.)

(b) \Rightarrow (c): same as (c) \Rightarrow (b), in reverse; (c) \Rightarrow (a): same as (a) \Rightarrow (c).

Problem 1.52

In Prob. 1.15 we found that $\nabla \cdot \mathbf{v}_a = 0$; in Prob. 1.18 we found that $\nabla \times \mathbf{v}_c = 0$. So

\mathbf{v}_c can be written as the gradient of a scalar; \mathbf{v}_a can be written as the curl of a vector.

(a) To find t :

$$(1) \frac{\partial t}{\partial x} = y^2 \Rightarrow t = y^2x + f(y, z)$$

$$(2) \frac{\partial t}{\partial y} = (2xy + z^2)$$

$$(3) \frac{\partial t}{\partial z} = 2yz$$

From (1) & (3) we get $\frac{\partial f}{\partial x} = 2yz \Rightarrow f = yz^2 + g(y) \Rightarrow t = y^2x + yz^2 + g(y)$, so $\frac{\partial t}{\partial y} = 2xy + z^2 + \frac{\partial g}{\partial y} = 2xy + z^2$ (from (2)) $\Rightarrow \frac{\partial g}{\partial y} = 0$. We may as well pick $g = 0$; then $t = xy^2 + yz^2$.

- (b) To find \mathbf{W} : $\frac{\partial W_x}{\partial y} - \frac{\partial W_y}{\partial x} = x^2$; $\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} = 3z^2x$; $\frac{\partial W_x}{\partial x} - \frac{\partial W_x}{\partial y} = -2xz$.
Pick $W_x = 0$; then

$$\begin{aligned}\frac{\partial W_z}{\partial x} &= -3xz^2 \Rightarrow W_z = -\frac{3}{2}x^2z^2 + f(y, z) \\ \frac{\partial W_y}{\partial x} &= -2xz \Rightarrow W_y = -x^2z + g(y, z).\end{aligned}$$

$\frac{\partial W_x}{\partial y} - \frac{\partial W_x}{\partial z} = \frac{\partial f}{\partial y} + x^2 - \frac{\partial g}{\partial z} = x^2 \Rightarrow \frac{\partial f}{\partial y} - \frac{\partial g}{\partial z} = 0$. May as well pick $f = g = 0$.

$$\boxed{\mathbf{W} = -x^2z \hat{y} - \frac{3}{2}x^2z^2 \hat{z}.$$

$$\text{Check: } \nabla \times \mathbf{W} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x^2z & -\frac{3}{2}x^2z^2 \end{vmatrix} = \hat{x} (x^2) + \hat{y} (3xz^2) + \hat{z} (-2xz). \checkmark$$

You can add any gradient (∇t) to \mathbf{W} without changing its curl, so this answer is far from unique. Some other solutions:

$$\mathbf{W} = xz^3 \hat{x} - x^2z \hat{y};$$

$$\mathbf{W} = (2xyz + xz^3) \hat{x} + x^2y \hat{z};$$

$$\mathbf{W} = xyz \hat{x} - \frac{1}{2}x^2z \hat{y} + \frac{1}{2}x^2 (y - 3z^2) \hat{z}.$$

Problem 1.53

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta r^2 \cos \phi) = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (-r^2 \cos \theta \sin \phi) \\ &= \frac{1}{r^2} 4r^3 \cos \theta + \frac{1}{r \sin \theta} \cos \theta r^2 \cos \phi + \frac{1}{r \sin \theta} (-r^2 \cos \theta \cos \phi) \\ &= \frac{r \cos \theta}{\sin \theta} [4 \sin \theta + \cos \phi - \cos \phi] = 4r \cos \theta.\end{aligned}$$

$$\begin{aligned}\int (\nabla \cdot \mathbf{v}) d\tau &= \int (4r \cos \theta) r^2 \sin \theta dr d\theta d\phi = 4 \int_0^R r^3 dr \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\phi \\ &= (R^4) \left(\frac{1}{2}\right) \left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}}.\end{aligned}$$

Surface consists of four parts:

- (1) Curved: $d\mathbf{a} = R^2 \sin \theta d\theta d\phi \hat{r}$; $r = R$. $\mathbf{v} \cdot d\mathbf{a} = (R^2 \cos \theta) (R^2 \sin \theta d\theta d\phi)$.

$$\int \mathbf{v} \cdot d\mathbf{a} = R^4 \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\phi = R^4 \left(\frac{1}{2}\right) \left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4}.$$

- (2) Left: $da = -r dr d\theta \hat{\phi}$; $\phi = 0$. $\mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \theta \sin \phi) (r dr d\theta) = 0$. $\int \mathbf{v} \cdot d\mathbf{a} = 0$.
 (3) Back: $da = r dr d\theta \hat{\phi}$; $\phi = \pi/2$. $\mathbf{v} \cdot d\mathbf{a} = (-r^2 \cos \theta \sin \phi) (r dr d\theta) = -r^3 \cos \theta dr d\theta$.

$$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^R r^3 dr \int_0^{\pi/2} \cos \theta d\theta = -\left(\frac{1}{4}R^4\right)(+1) = -\frac{1}{4}R^4.$$

- (4) Bottom: $da = r \sin \theta dr d\phi \hat{\theta}$; $\theta = \pi/2$. $\mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \phi) (r dr d\phi)$.

$$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^R r^3 dr \int_0^{\pi/2} \cos \phi d\phi = \frac{1}{4}R^4.$$

Total: $\oint \mathbf{v} \cdot d\mathbf{a} = \pi R^4/4 + 0 - \frac{1}{4}R^4 + \frac{1}{4}R^4 = \frac{\pi R^4}{4}$. ✓

Problem 1.54

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} = \hat{z}(b-a). \quad \text{So } \int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = (b-a)\pi R^2.$$

$\mathbf{v} \cdot d\mathbf{l} = (ay \hat{x} + bx \hat{y}) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z}) = ay dx + bx dy$; $x^2 + y^2 = R^2 \Rightarrow 2x dx + 2y dy = 0$,
 so $dy = -(x/y) dx$. So $\mathbf{v} \cdot d\mathbf{l} = ay dx + bx(-x/y) dx = \frac{1}{y}(ay^2 - bx^2) dx$.

For the "upper" semicircle, $y = \sqrt{R^2 - x^2}$, so $\mathbf{v} \cdot d\mathbf{l} = \frac{a(R^2 - x^2) - bx^2}{\sqrt{R^2 - x^2}} dx$.

$$\begin{aligned} \int \mathbf{v} \cdot d\mathbf{l} &= \int_R^{-R} \frac{aR^2 - (a+b)x^2}{\sqrt{R^2 - x^2}} dx = \left\{ aR^2 \sin^{-1}\left(\frac{x}{R}\right) - (a+b) \left[-\frac{x}{2} \sqrt{R^2 - x^2} + \frac{R^2}{2} \sin^{-1}\left(\frac{x}{R}\right) \right] \right\} \Big|_{+R}^{-R} \\ &= \frac{1}{2}R^2(a-b) \sin^{-1}(x/R) \Big|_{+R}^{-R} = \frac{1}{2}R^2(a-b) (\sin^{-1}(-1) - \sin^{-1}(+1)) = \frac{1}{2}R^2(a-b) \left(-\frac{\pi}{2} - \frac{\pi}{2}\right) \\ &= \frac{1}{2}\pi R^2(b-a). \end{aligned}$$

And the same for the lower semicircle (y changes sign, but the limits on the integral are reversed) so

$$\oint \mathbf{v} \cdot d\mathbf{l} = \pi R^2(b-a). \quad \checkmark$$

Problem 1.55

- (1) $x = z = 0$; $dx = dz = 0$; $y : 0 \rightarrow 1$. $\mathbf{v} \cdot d\mathbf{l} = (y + 3x) dy = y dy$.

$$\int_0^1 \mathbf{v} \cdot d\mathbf{l} = \int_0^1 y dy = \frac{1}{2}.$$

- (2) $x = 0$; $z = 2 - 2y$; $dz = -2 dy$; $y : 1 \rightarrow 0$. $\mathbf{v} \cdot d\mathbf{l} = (y + 3x) dy + 6 dz = y dy - 12 dy = (y - 12) dy$.

$$\int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 (y - 12) dy = -\left(\frac{1}{2} - 12\right) = -\frac{1}{2} + 12.$$

- (3) $x = y = 0$; $dx = dy = 0$; $z : 2 \rightarrow 0$. $\mathbf{v} \cdot d\mathbf{l} = 6 dz$;

$$\int \mathbf{v} \cdot d\mathbf{l} = \int_2^0 6 dz = -12.$$

$$\text{Total: } \oint \mathbf{v} \cdot d\mathbf{l} = \frac{1}{2} - \frac{1}{2} + 12 - 12 = \boxed{0}.$$

Meanwhile, Stokes' theorem says $\oint \mathbf{v} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$. Here $d\mathbf{a} = dy dz \hat{x}$, so all we need is $(\nabla \times \mathbf{v})_x = \frac{\partial}{\partial y}(6) - \frac{\partial}{\partial z}(y + 3z) = 0$. Therefore $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$. \checkmark

Problem 1.56

Start at the origin.

$$(1) \theta = \frac{\pi}{2}, \phi = 0; r: 0 \rightarrow 1. \quad \mathbf{v} \cdot d\mathbf{l} = (r \cos^2 \theta) (dr) = 0. \quad \int \mathbf{v} \cdot d\mathbf{l} = 0.$$

$$(2) r = 1, \theta = \frac{\pi}{2}; \phi: 0 \rightarrow \pi/2. \quad \mathbf{v} \cdot d\mathbf{l} = (3r)(r \sin \theta d\phi) = 3d\phi. \quad \int \mathbf{v} \cdot d\mathbf{l} = 3 \int_0^{\pi/2} d\phi = \frac{3\pi}{2}.$$

$$(3) \phi = \frac{\pi}{2}; r \sin \theta = y = 1, \text{ so } r = \frac{1}{\sin \theta}, dr = \frac{-1}{\sin^2 \theta} \cos \theta d\theta, \theta: \frac{\pi}{2} \rightarrow \frac{\pi}{4}.$$

$$\begin{aligned} \mathbf{v} \cdot d\mathbf{l} &= (r \cos^2 \theta) (dr) - (r \cos \theta \sin \theta)(r d\theta) = \frac{\cos^2 \theta}{\sin \theta} \left(-\frac{\cos \theta}{\sin^2 \theta} \right) d\theta - \frac{\cos \theta \sin \theta}{\sin^2 \theta} d\theta \\ &= -\left(\frac{\cos^3 \theta}{\sin^3 \theta} + \frac{\cos \theta}{\sin \theta} \right) d\theta = -\frac{\cos \theta}{\sin \theta} \left(\frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \theta} \right) d\theta = -\frac{\cos \theta}{\sin^3 \theta} d\theta. \end{aligned}$$

Therefore

$$\int \mathbf{v} \cdot d\mathbf{l} = -\int_{\pi/2}^{\pi/4} \frac{\cos \theta}{\sin^3 \theta} d\theta = \frac{1}{2 \sin^2 \theta} \Big|_{\pi/2}^{\pi/4} = \frac{1}{2 \cdot (1/2)} - \frac{1}{2 \cdot (1)} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$(4) \theta = \frac{\pi}{4}, \phi = \frac{\pi}{2}; r: \sqrt{2} \rightarrow 0. \quad \mathbf{v} \cdot d\mathbf{l} = (r \cos^2 \theta) (dr) = \frac{1}{2} r dr.$$

$$\int \mathbf{v} \cdot d\mathbf{l} = \frac{1}{2} \int_{\sqrt{2}}^0 r dr = \frac{1}{2} \frac{r^2}{2} \Big|_{\sqrt{2}}^0 = -\frac{1}{4} \cdot 2 = -\frac{1}{2}.$$

Total:

$$\oint \mathbf{v} \cdot d\mathbf{l} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \boxed{\frac{3\pi}{2}}.$$

Stokes' theorem says this should equal $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$

$$\begin{aligned} \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta 3r) - \frac{\partial}{\partial \phi} (-r \sin \theta \cos \theta) \right] \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} (r \cos^2 \theta) - \frac{\partial}{\partial r} (r 3r) \right] \hat{\boldsymbol{\theta}} \\ &\quad + \frac{1}{r} \left[\frac{\partial}{\partial r} (-r r \cos \theta \sin \theta) - \frac{\partial}{\partial \theta} (r \cos^2 \theta) \right] \hat{\boldsymbol{\phi}} \\ &= \frac{1}{r \sin \theta} [3r \cos \theta] \hat{\mathbf{r}} + \frac{1}{r} [-6r] \hat{\boldsymbol{\theta}} + \frac{1}{r} [-2r \cos \theta \sin \theta + 2r \cos \theta \sin \theta] \hat{\boldsymbol{\phi}} \\ &= 3 \cot \theta \hat{\mathbf{r}} - 6 \hat{\boldsymbol{\theta}}. \end{aligned}$$

$$(1) \text{ Back face: } d\mathbf{a} = -r dr d\theta \hat{\boldsymbol{\phi}}; (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0. \quad \int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0.$$

$$(2) \text{ Bottom: } d\mathbf{a} = -r \sin \theta dr d\phi \hat{\boldsymbol{\theta}}; (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 6r \sin \theta dr d\phi. \quad \theta = \frac{\pi}{2}, \text{ so } (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 6r dr d\phi$$

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 6r dr \int_0^{\pi/2} d\phi = 6 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{2}. \quad \checkmark$$

Problem 1.57

$$\mathbf{v} \cdot d\mathbf{l} = y \, dz.$$

(1) *Left side:* $z = a - x$; $dz = -dx$; $y = 0$. Therefore $\int \mathbf{v} \cdot d\mathbf{l} = 0$.

(2) *Bottom:* $dz = 0$. Therefore $\int \mathbf{v} \cdot d\mathbf{l} = 0$.

(3) *Back:* $z = a - \frac{1}{2}y$; $dz = -1/2 \, dy$; $y : 2a \rightarrow 0$. $\int \mathbf{v} \cdot d\mathbf{l} = \int_{2a}^0 y \, (-\frac{1}{2} \, dy) = -\frac{1}{2} \frac{y^2}{2} \Big|_{2a}^0 = -\frac{4a^2}{4} = \boxed{a^2}$.

Meanwhile, $\nabla \times \mathbf{v} = \hat{\mathbf{x}}$, so $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ is the projection of this surface on the xy plane $= \frac{1}{2} \cdot a \cdot 2a = a^2$. \checkmark

Problem 1.58

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2 \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \, 4r^2 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (r^2 \tan \theta) \\ &= \frac{1}{r^2} 4r^3 \sin \theta + \frac{1}{r \sin \theta} 4r^2 (\cos^2 \theta - \sin^2 \theta) = \frac{4r}{\sin \theta} (\sin^2 \theta + \cos^2 \theta - \sin^2 \theta) \\ &= 4r \frac{\cos^2 \theta}{\sin \theta}. \end{aligned}$$

$$\begin{aligned} \int (\nabla \cdot \mathbf{v}) \, dr &= \int \left(4r \frac{\cos^2 \theta}{\sin \theta} \right) (r^2 \sin \theta \, dr \, d\theta \, d\phi) = \int_0^R 4r^3 \, dr \int_0^{\pi/6} \cos^2 \theta \, d\theta \int_0^{2\pi} d\phi = (R^4) (2\pi) \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/6} \\ &= 2\pi R^4 \left(\frac{\pi}{12} + \frac{\sin 60^\circ}{4} \right) = \frac{\pi R^4}{6} \left(\pi + 3 \frac{\sqrt{3}}{2} \right) = \boxed{\frac{\pi R^4}{12} (2\pi + 3\sqrt{3})}. \end{aligned}$$

Surface consists of two parts:

(1) *The ice cream:* $r = R$; $\phi : 0 \rightarrow 2\pi$; $\theta : 0 \rightarrow \pi/6$; $d\mathbf{a} = R^2 \sin \theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$; $\mathbf{v} \cdot d\mathbf{a} = (R^2 \sin \theta) (R^2 \sin \theta \, d\theta \, d\phi) = R^4 \sin^2 \theta \, d\theta \, d\phi$.

$$\int \mathbf{v} \cdot d\mathbf{a} = R^4 \int_0^{\pi/6} \sin^2 \theta \, d\theta \int_0^{2\pi} d\phi = (R^4) (2\pi) \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/6} = 2\pi R^4 \left(\frac{\pi}{12} - \frac{1}{4} \sin 60^\circ \right) = \frac{\pi R^4}{6} \left(\pi - 3 \frac{\sqrt{3}}{2} \right)$$

(2) *The cone:* $\theta = \frac{\pi}{6}$; $\phi : 0 \rightarrow 2\pi$; $r : 0 \rightarrow R$; $d\mathbf{a} = r \sin \theta \, dr \, d\phi \, \hat{\boldsymbol{\theta}} = \frac{\sqrt{3}}{2} r \, dr \, d\phi \, \hat{\boldsymbol{\theta}}$; $\mathbf{v} \cdot d\mathbf{a} = \sqrt{3} r^3 \, dr \, d\phi$

$$\int \mathbf{v} \cdot d\mathbf{a} = \sqrt{3} \int_0^R r^3 \, dr \int_0^{2\pi} d\phi = \sqrt{3} \cdot \frac{R^4}{4} \cdot 2\pi = \frac{\sqrt{3}}{2} \pi R^4.$$

Therefore $\int \mathbf{v} \cdot d\mathbf{a} = \frac{\pi R^4}{2} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3} \right) = \frac{\pi R^4}{12} (2\pi + 3\sqrt{3})$. \checkmark

Problem 1.59

(a) Corollary 2 says $\oint (\nabla T) \cdot d\mathbf{l} = 0$. Stokes' theorem says $\oint (\nabla T) \cdot d\mathbf{l} = \int [\nabla \times (\nabla T)] \cdot d\mathbf{a}$. So $\int [\nabla \times (\nabla T)] \cdot d\mathbf{a} = 0$, and since this is true for *any* surface, the integrand must vanish: $\nabla \times (\nabla T) = 0$, confirming Eq. 1.44.

(b) Corollary 2 says $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$. Divergence theorem says $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int \nabla \cdot (\nabla \times \mathbf{v}) d\tau$. So $\int \nabla \cdot (\nabla \times \mathbf{v}) d\tau = 0$, and since this is true for *any* volume, the integrand must vanish: $\nabla \cdot (\nabla \times \mathbf{v}) = 0$, confirming Eq. 1.46.

Problem 1.60

(a) Divergence theorem: $\oint \mathbf{v} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{v}) d\tau$. Let $\mathbf{v} = cT$, where c is a constant vector. Using product rule #5 in front cover: $\nabla \cdot \mathbf{v} = \nabla \cdot (cT) = T(\nabla \cdot c) + c \cdot (\nabla T)$. But c is constant so $\nabla \cdot c = 0$. Therefore we have: $\oint c \cdot (\nabla T) d\tau = \int Tc \cdot d\mathbf{a}$. Since c is constant, take it outside the integrals: $c \cdot \int \nabla T d\tau = c \cdot \int T d\mathbf{a}$. But c is *any* constant vector—in particular, it could be \hat{x} , or \hat{y} , or \hat{z} —so each *component* of the integral on left equals corresponding component on the right, and hence

$$\int \nabla T d\tau = \int T d\mathbf{a}. \quad \text{qed}$$

(b) Let $\mathbf{v} \rightarrow (\mathbf{v} \times c)$ in divergence theorem. Then $\int \nabla \cdot (\mathbf{v} \times c) d\tau = \int (\mathbf{v} \times c) \cdot d\mathbf{a}$. Product rule #6 $\Rightarrow \nabla \cdot (\mathbf{v} \times c) = c \cdot (\nabla \times \mathbf{v}) - \mathbf{v} \cdot (\nabla \times c) = c \cdot (\nabla \times \mathbf{v})$. (Note: $\nabla \times c = 0$, since c is constant.) Meanwhile vector identity (1) says $da \cdot (\mathbf{v} \times c) = c \cdot (da \times \mathbf{v}) = -c \cdot (\mathbf{v} \times da)$. Thus $\int c \cdot (\nabla \times \mathbf{v}) d\tau = -\int c \cdot (\mathbf{v} \times da)$. Take c outside, and again let c be \hat{x} , \hat{y} , \hat{z} then:

$$\int (\nabla \times \mathbf{v}) d\tau = -\int \mathbf{v} \times d\mathbf{a}. \quad \text{qed}$$

(c) Let $\mathbf{v} = T\nabla U$ in divergence theorem: $\int \nabla \cdot (T\nabla U) d\tau = \int T\nabla U \cdot d\mathbf{a}$. Product rule #(5) $\Rightarrow \nabla \cdot (T\nabla U) = T\nabla \cdot (\nabla U) + (\nabla U) \cdot (\nabla T) = T\nabla^2 U + (\nabla U) \cdot (\nabla T)$. Therefore

$$\int (T\nabla^2 U + (\nabla U) \cdot (\nabla T)) d\tau = \int (T\nabla U) \cdot d\mathbf{a}. \quad \text{qed}$$

(d) Rewrite (c) with $T \leftrightarrow U$: $\int (U\nabla^2 T + (\nabla T) \cdot (\nabla U)) d\tau = \int (U\nabla T) \cdot d\mathbf{a}$. Subtract this from (c), noting that the $(\nabla U) \cdot (\nabla T)$ terms cancel:

$$\int (T\nabla^2 U - U\nabla^2 T) d\tau = \int (T\nabla U - U\nabla T) \cdot d\mathbf{a}. \quad \text{qed}$$

(e) Stoke's theorem: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint \mathbf{v} \cdot d\mathbf{l}$. Let $\mathbf{v} = cT$. By Product Rule #(7): $\nabla \times (cT) = T(\nabla \times c) - c \times (\nabla T) = -c \times (\nabla T)$ (since c is constant). Therefore, $-\int (c \times (\nabla T)) \cdot d\mathbf{a} = \oint Tc \cdot d\mathbf{l}$. Use vector identity #1 to rewrite the first term $(c \times (\nabla T)) \cdot d\mathbf{a} = c \cdot (\nabla T \times d\mathbf{a})$. So $-\int c \cdot (\nabla T \times d\mathbf{a}) = \oint Tc \cdot d\mathbf{l}$. Pull c outside, and let $c \rightarrow \hat{x}$, \hat{y} , and \hat{z} to prove:

$$\int \nabla T \times d\mathbf{a} = -\oint T d\mathbf{l}. \quad \text{qed}$$

Problem 1.61

(a) $da = R^2 \sin \theta d\theta d\phi \hat{r}$. Let the surface be the northern hemisphere. The \hat{x} and \hat{y} components clearly integrate to zero, and the \hat{z} component of \hat{r} is $\cos \theta$, so

$$\mathbf{a} = \int R^2 \sin \theta \cos \theta d\theta d\phi \hat{z} = 2\pi R^2 \hat{z} \int_0^{\pi/2} \sin \theta \cos \theta d\theta = 2\pi R^2 \hat{z} \left. \frac{\sin^2 \theta}{2} \right|_0^{\pi/2} = \boxed{\pi R^2 \hat{z}}.$$

(b) Let $T = 1$ in Prob. 1.60(a). Then $\nabla T = 0$, so $\oint \mathbf{v} \cdot d\mathbf{l} = 0$. qed.

(c) This follows from (b). For suppose $\mathbf{a}_1 \neq \mathbf{a}_2$; then if you put them together to make a closed surface, $\oint d\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2 \neq 0$.

(d) For one such triangle, $d\mathbf{a} = \frac{1}{2}(\mathbf{r} \times d\mathbf{l})$ (since $\mathbf{r} \times d\mathbf{l}$ is the area of the parallelogram, and the direction is perpendicular to the surface), so for the entire conical surface, $\mathbf{a} = \frac{1}{2} \oint \mathbf{r} \times d\mathbf{l}$.

(e) Let $T = \mathbf{c} \cdot \mathbf{r}$, and use product rule #4: $\nabla T = \nabla(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c} \times (\nabla \times \mathbf{r}) + (\mathbf{c} \cdot \nabla)\mathbf{r}$. But $\nabla \times \mathbf{r} = \mathbf{0}$, and $(\mathbf{c} \cdot \nabla)\mathbf{r} = (c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z})(x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}) = c_x \hat{\mathbf{x}} + c_y \hat{\mathbf{y}} + c_z \hat{\mathbf{z}} = \mathbf{c}$. So Prob. 1.60(e) says

$$\oint T d\mathbf{l} = \oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} = - \int (\nabla T) \times d\mathbf{a} = - \int \mathbf{c} \times d\mathbf{a} = -\mathbf{c} \times \int d\mathbf{a} = -\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{c}. \quad \text{qed}$$

Problem 1.62

(1)

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (r) = \boxed{\frac{1}{r^2}}$$

For a sphere of radius R :

$$\left. \begin{aligned} \int \mathbf{v} \cdot d\mathbf{a} &= \int \left(\frac{1}{R} \hat{\mathbf{r}} \right) \cdot (R^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}) = R \int \sin \theta d\theta d\phi = 4\pi R. \\ \int (\nabla \cdot \mathbf{v}) d\tau &= \int \left(\frac{1}{r^2} \right) (r^2 \sin \theta dr d\theta d\phi) = \left(\int_0^R dr \right) \left(\int \sin \theta d\theta d\phi \right) = 4\pi R. \end{aligned} \right\} \text{So divergence theorem checks.}$$

Evidently there is *no* delta function at the origin.

$$\nabla \times (r^n \hat{\mathbf{r}}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^n) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^{n+2}) = \frac{1}{r^2} (n+2) r^{n+1} = \boxed{(n+2)r^{n-1}}$$

(except for $n = -2$, for which we already know (Eq. 1.99) that the divergence is $4\pi\delta^3(\mathbf{r})$).

- (2) *Geometrically*, it should be zero. Likewise, the curl in the spherical coordinates obviously gives zero. To be certain there is no lurking delta function here, we integrate over a sphere of radius R , using Prob. 1.60(b): If $\nabla \times (r^n \hat{\mathbf{r}}) = \mathbf{0}$, then $\int (\nabla \times \mathbf{v}) d\tau = 0 \stackrel{?}{=} - \oint \mathbf{v} \times d\mathbf{a}$. But $\mathbf{v} = r^n \hat{\mathbf{r}}$ and $d\mathbf{a} = R^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}$ are both in the $\hat{\mathbf{r}}$ directions, so $\mathbf{v} \times d\mathbf{a} = 0$. \checkmark