COMPLEX NUMBERS
AND
QUADRATIC EQUATIONS

WITH

COMPLETE SOLUTIONS

Complex Numbers

A number of the form $a + ib$, where a and b are real numbers, is defined to be a complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4+i\left(\frac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by Re z and b is called the *imaginary part* denoted by Im z *of the complex number* z. For example, if $z = 2 + i5$, then Re $z = 2$ and Im $z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Algebra of Complex Numbers

Addition of two complex numbers Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + c) + i\,(b + d)$, which is again a complex number.

For example, $(2 + i3) + (-6 + i5) = (2 - 6) + i\,(3 + 5) = -4 + i\,8$

The addition of complex numbers satisfy the following properties:

(i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) *The commutative law* For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

(iii) *The associative law* For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iv) *The existence of additive identity* There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number*, such that, for every complex number z, $z + 0 = z$.

(v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of* z. We observe that $z + (-z) = 0$ (the additive identity).

Difference of two complex numbers Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows:

$$z_1 - z_2 = z_1 + (-z_2).$$

For example, $(6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and $(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

Multiplication of two complex numbers Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$z_1 z_2 = (ac - bd) + i(ad + bc)$

For example, $(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law** For any two complex numbers z_1 and z_2,

$$z_1 z_2 = z_2 z_1.$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i\,0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi(a \neq 0, b \neq 0)$, we have the complex number

$\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that

$z.\frac{1}{z} = 1$ (the multiplicative identity).

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3,

(a) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

Division of two complex numbers Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

Then $\frac{z_1}{z_2} = \left((6+3i)\times\frac{1}{2-i}\right) = (6+3i)\left(\frac{2}{2^2+(-1)^2} + i\frac{-(-1)}{2^2+(-1)^2}\right)$

$$= (6+3i)\left(\frac{2+i}{5}\right) = \frac{1}{5}\left[12-3+i(6+6)\right] = \frac{1}{5}(9+12i)$$

Power of i we know that

$$i^3 = i^2 i = (-1)\, i = -i, \qquad i^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

$$i^5 = \left(i^2\right)^2 i = (-1)^2\, i = i, \qquad i^6 = \left(i^2\right)^3 = (-1)^3 = -1, \text{ etc.}$$

Also, we have $\quad i^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

Identities We prove the following identity

$(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1 z_2$, for all complex numbers z_1 and z_2.

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)\,(z_1 + z_2)$,

$= (z_1 + z_2)\, z_1 + (z_1 + z_2)\, z_2$ (Distributive law)

$= z_1^2 + z_2 z_1 + z_1 z_2 + z_2^2$ (Distributive law)

$= z_1^2 + z_1 z_2 + z_1 z_2 + z_2^2$ (Commutative law of multiplication)

$= z_1^2 + 2z_1 z_2 + z_2^2$

Similarly, we can prove the following identities:

(i) $(z_1 - z_2)^2 = z_1^2 - 2\, z_1\, z_2 + z_2^2$

(ii) $(z_1 + z_2)^3 = z_1^3 + 3\, z_1^2 z_2 + 3 z_1 z_2^2 + z_2^3$

(iii) $(z_1 - z_2)^3 = z_1^3 - 3 z_1^2 z_2 + 3 z_1 z_2^2 - z_2^3$

(iv) $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z, denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2 + b^2}$, i.e., $|z| = \sqrt{a^2 + b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $|3+i| = \sqrt{3^2+1^2} = \sqrt{10}$, $|2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$,

and $\overline{3+i} = 3-i$, $\overline{2-5i} = 2+5i$, $\overline{-3i-5} = 3i-5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2} = \frac{a-ib}{a^2+b^2} = \frac{\overline{z}}{|z|^2}$$

or $z\,\overline{z} = |z|^2$

Furthermore, the following results can easily be derived.

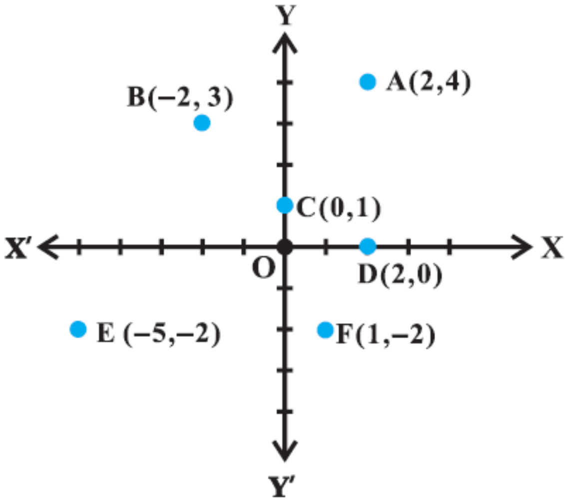
For any two compex numbers z_1 and z_2, we have

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (iv) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$ (v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ provided $z_2 \neq 0$.

Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point P(x, y) in the XY-plane and vice-versa.

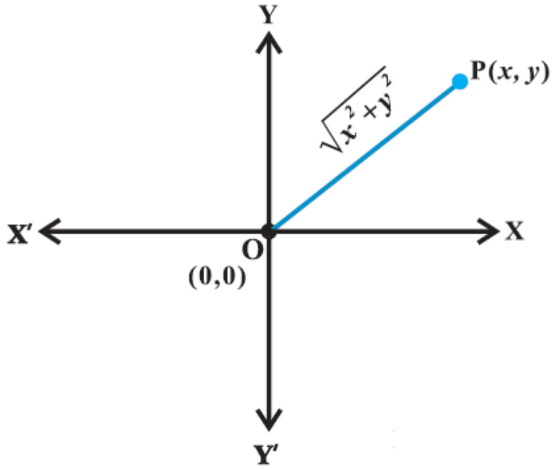


Fig

Some complex numbers such as $2 + 4i$, $-2 + 3i$, $0 + 1i$, $2 + 0i$, $-5 -2i$ and $1 - 2i$ which correspond to the ordered pairs (2, 4), (− 2, 3), (0, 1), (2, 0), (−5, −2), and (1, − 2), respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.

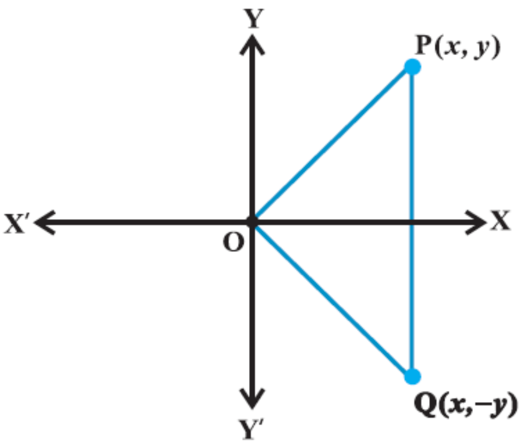
Obviously, in the Argand plane, the modulus of the complex number $x + iy = \sqrt{x^2 + y^2}$ is the distance between the point P(x, y) and the origin O (0, 0) (Fig 5.2). The points on the x-axis corresponds to the complex numbers of the form $a + i\,0$ and the points on the y-axis corresponds to the complex numbers of the form



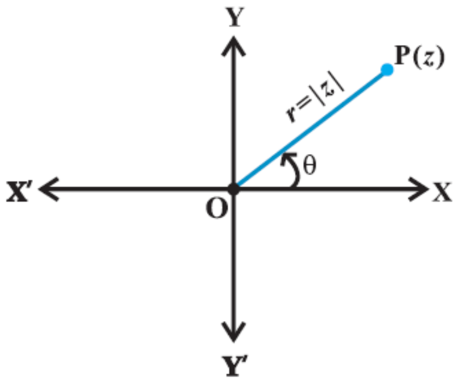
$0 + i\,b$. The x-axis and y-axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number $z = x + iy$ and its conjugate $z = x - iy$ in the Argand plane are, respectively, the points P (x, y) and Q (x, $-y$).

Geometrically, the point (x, $-y$) is the mirror image of the point (x, y) on the real axis



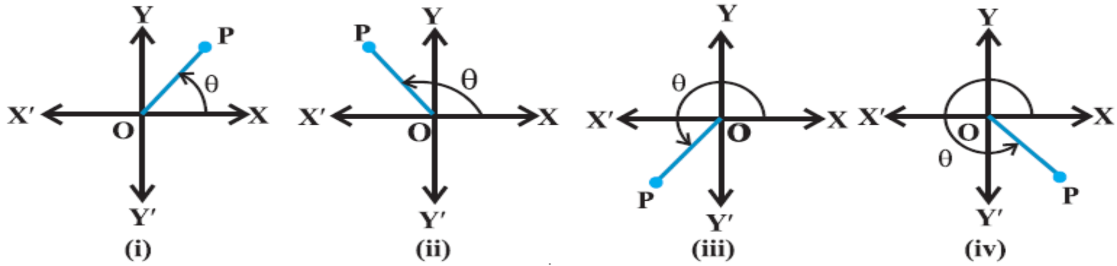
Polar representation of a complex number Let the point P represent the non-zero complex number $z = x + iy$. Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis (Fig 5.4).



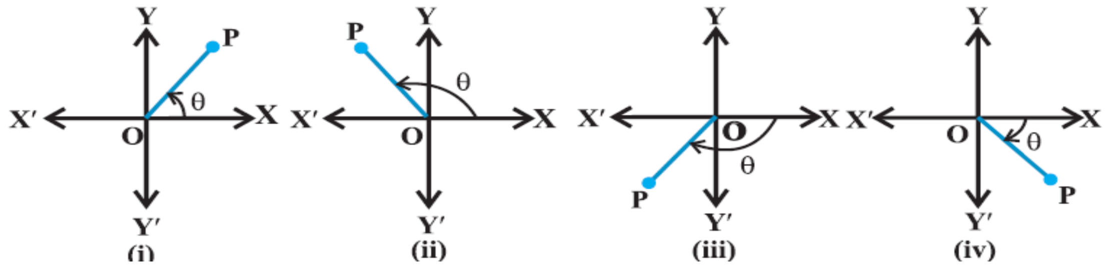
We may note that the point P is uniquely determined by the ordered pair of real numbers (r, θ), called the *polar coordinates of the point* P. We consider the origin as the pole and the positive direction of the x axis as the initial line.

We have, $x = r \cos \theta$, $y = r \sin \theta$ and therefore, $z = r (\cos \theta + i \sin \theta)$. The latter is said to be the *polar form of the complex number*. Here $r = \sqrt{x^2 + y^2} = |z|$ is the modulus of z and θ is called the argument (or amplitude) of z which is denoted by arg z.

For any complex number $z \neq 0$, there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be such an interval. We shall take the value of θ such that $-\pi < \theta \leq \pi$, called **principal argument** of z and is denoted by arg z, unless specified otherwise.



$(0 \leq \theta < 2\pi)$



$(-\pi < \theta \leq \pi)$

Quadratic Equations

Let us consider the following quadratic equation:

$ax^2 + bx + c = 0$ with real coefficients a, b, c and $a \neq 0$.

Also, let us assume that the $b^2 - 4ac < 0$.

Now, we know that we can find the square root of negative real numbers in the set of complex numbers. Therefore, the solutions to the above equation are available in the set of complex numbers which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

Mathematics

(Chapter – 5) (Complex Numbers and Quadratic Equations)

(Class – XI)

Exercise 5.1

Question 1:

Express the given complex number in the form *a* + *ib*: $(5i)\left(-\frac{3}{5}i\right)$

Answer 1:

$$\begin{aligned}(5i)\left(\frac{-3}{5}i\right) &= -5\times\frac{3}{5}\times i\times i\\ &= -3i^2\\ &= -3(-1) \qquad [i^2=-1]\\ &= 3\end{aligned}$$

Question 2:

Express the given complex number in the form *a* + *ib*: $i^9 + i^{19}$

Answer 2:

$$\begin{aligned}i^9+i^{19} &= i^{4\times2+1}+i^{4\times4+3}\\ &= \left(i^4\right)^2\cdot i+\left(i^4\right)^4\cdot i^3\\ &= 1\times i+1\times(-i) \qquad [i^4=1,\ i^3=-i]\\ &= i+(-i)\\ &= 0\end{aligned}$$

Question 3:

Express the given complex number in the form *a* + *ib*: i^{-39}

Answer 3:

$$i^{-39} = i^{-4\times9-3} = \left(i^4\right)^{-9} \cdot i^{-3}$$

$$= (1)^{-9} \cdot i^{-3} \qquad \left[i^4 = 1\right]$$

$$= \frac{1}{i^3} = \frac{1}{-i} \qquad \left[i^3 = -i\right]$$

$$= \frac{-1}{i} \times \frac{i}{i}$$

$$= \frac{-i}{i^2} = \frac{-i}{-1} = i \qquad \left[i^2 = -1\right]$$

Question 4:

Express the given complex number in the form *a* + *ib*:

3(7 + *i*7) + *i*(7 + *i*7)

Answer 4:

$$3(7+i7)+i(7+i7) = 21+21i+7i+7i^2$$

$$= 21+28i+7\times(-1) \qquad \left[\because i^2 = -1\right]$$

$$= 14+28i$$

Question 5:

Express the given complex number in the form *a* + *ib*: (1 − *i*) − (−1 + *i*6)

Answer 5:

$$(1-i)-(-1+i6) = 1-i+1-6i$$

$$= 2-7i$$

Question 6:

Express the given complex number in the form *a* + *ib*: $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

Answer 6:

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$$
$$=\frac{1}{5}+\frac{2}{5}i-4-\frac{5}{2}i$$
$$=\left(\frac{1}{5}-4\right)+i\left(\frac{2}{5}-\frac{5}{2}\right)$$
$$=\frac{-19}{5}+i\left(\frac{-21}{10}\right)$$
$$=\frac{-19}{5}-\frac{21}{10}i$$

Question 7:

Express the given complex number in the form *a* + *ib*:

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$$

Answer 7:

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(\frac{-4}{3}+i\right)$$
$$=\frac{1}{3}+\frac{7}{3}i+4+\frac{1}{3}i+\frac{4}{3}-i$$
$$=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right)$$
$$=\frac{17}{3}+i\frac{5}{3}$$

Question 8:

Express the given complex number in the form *a* + *ib*: $(1 - i)^4$

Answer 8:

$$(1-i)^4 = \left[(1-i)^2\right]^2$$
$$= \left[1^2 + i^2 - 2i\right]^2$$
$$= [1-1-2i]^2$$
$$= (-2i)^2$$
$$= (-2i)\times(-2i)$$
$$= 4i^2 = -4 \qquad \left[i^2 = -1\right]$$

Question 9:

Express the given complex number in the form *a* + *ib*: $\left(\frac{1}{3}+3i\right)^3$

Answer 9:

$$\left(\frac{1}{3}+3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3\left(\frac{1}{3}\right)(3i)\left(\frac{1}{3}+3i\right)$$
$$= \frac{1}{27} + 27i^3 + 3i\left(\frac{1}{3}+3i\right)$$
$$= \frac{1}{27} + 27(-i) + i + 9i^2 \qquad \left[i^3 = -i\right]$$
$$= \frac{1}{27} - 27i + i - 9 \qquad \left[i^2 = -1\right]$$
$$= \left(\frac{1}{27} - 9\right) + i(-27+1)$$
$$= \frac{-242}{27} - 26i$$

Question 10:

Express the given complex number in the form *a* + *ib*: $\left(-2-\frac{1}{3}i\right)^3$

Answer 10:

$$\left(-2-\frac{1}{3}i\right)^3=(-1)^3\left(2+\frac{1}{3}i\right)^3$$

$$=-\left[2^3+\left(\frac{i}{3}\right)^3+3(2)\left(\frac{i}{3}\right)\left(2+\frac{i}{3}\right)\right]$$

$$=-\left[8+\frac{i^3}{27}+2i\left(2+\frac{i}{3}\right)\right]$$

$$=-\left[8-\frac{i}{27}+4i+\frac{2i^2}{3}\right] \qquad \left[i^3=-i\right]$$

$$=-\left[8-\frac{i}{27}+4i-\frac{2}{3}\right] \qquad \left[i^2=-1\right]$$

$$=-\left[\frac{22}{3}+\frac{107i}{27}\right]$$

$$=-\frac{22}{3}-\frac{107}{27}i$$

Question 11:

Find the multiplicative inverse of the complex number 4 – 3*i*.

Answer 11:

Let *z* = 4 – 3*i*

Then,

$\bar{z}$ = 4 + 3*i* and $|z|^2=4^2+(-3)^2=16+9=25$

Therefore, the multiplicative inverse of 4 – 3*i* is given by

$$z^{-1}=\frac{\bar{z}}{|z|^2}=\frac{4+3i}{25}=\frac{4}{25}+\frac{3}{25}i$$

Question 12:

Find the multiplicative inverse of the complex number $\sqrt{5}+3i$

Answer 12:

Let $z = \sqrt{5}+3i$

Then, $\bar{z} = \sqrt{5}-3i$ and $|z|^2 = \left(\sqrt{5}\right)^2 + 3^2 = 5+9 = 14$

Therefore, the multiplicative inverse of $\sqrt{5}+3i$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5}-3i}{14} = \frac{\sqrt{5}}{14} - \frac{3i}{14}$$

Question 13:

Find the multiplicative inverse of the complex number $-i$

Answer 13:

Let $z = -i$

Then, $\bar{z} = i$ and $|z|^2 = 1^2 = 1$

Therefore, the multiplicative inverse of $-i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i$$

Question 14:

Express the following expression in the form of *a* + *ib*.

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

Answer 14:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

$$=\frac{(3)^2-\left(i\sqrt{5}\right)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i} \qquad \left[(a+b)(a-b)=a^2-b^2\right]$$

$$=\frac{9-5i^2}{2\sqrt{2}i}$$

$$=\frac{9-5(-1)}{2\sqrt{2}i} \qquad \left[i^2=-1\right]$$

$$=\frac{9+5}{2\sqrt{2}i}\times\frac{i}{i}$$

$$=\frac{14i}{2\sqrt{2}i^2}$$

$$=\frac{14i}{2\sqrt{2}(-1)}$$

$$=\frac{-7i}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{-7\sqrt{2}i}{2}$$

Mathematics

(Chapter – 5) (Complex Numbers and Quadratic Equations)

(Class – XI)

Exercise 5.2

Question 1:

Find the modulus and the argument of the complex number $z = -1 - i\sqrt{3}$

Answer 1:

$z = -1 - i\sqrt{3}$

Let $r\cos\theta = -1$ and $r\sin\theta = -\sqrt{3}$

On squaring and adding, we obtain

$$(r\cos\theta)^2 + (r\sin\theta)^2 = (-1)^2 + \left(-\sqrt{3}\right)^2$$

$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 1 + 3$$

$$\Rightarrow r^2 = 4 \qquad \left[\cos^2\theta + \sin^2\theta = 1\right]$$

$$\Rightarrow r = \sqrt{4} = 2 \qquad [\text{Conventionally, } r > 0]$$

$$\therefore \text{Modulus} = 2$$

$$\therefore\ 2\cos\theta = -1 \text{ and } 2\sin\theta = -\sqrt{3}$$

$$\Rightarrow \cos\theta = \frac{-1}{2} \text{ and } \sin\theta = \frac{-\sqrt{3}}{2}$$

Since both the values of sin θ and cos θ are negative and sinθ and cosθ are negative in

III quadrant,

$$\text{Argument} = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$$

Thus, the modulus and argument of the complex number $-1 - \sqrt{3}i$ are 2 and $\frac{-2\pi}{3}$ respectively.

Question 2:

Find the modulus and the argument of the complex number $z=-\sqrt{3}+i$

Answer 2:

$z=-\sqrt{3}+i$

Let $r\cos\theta=-\sqrt{3}$ and $r\sin\theta=1$

On squaring and adding, we obtain

$$r^2\cos^2\theta+r^2\sin^2\theta=\left(-\sqrt{3}\right)^2+1^2$$

$$\Rightarrow r^2=3+1=4 \qquad \left[\cos^2\theta+\sin^2\theta=1\right]$$

$$\Rightarrow r=\sqrt{4}=2 \qquad \left[\text{Conventionally, } r>0\right]$$

$$\therefore \text{Modulus }=2$$

$$\therefore\ 2\cos\theta=-\sqrt{3} \text{ and } 2\sin\theta=1$$

$$\Rightarrow \cos\theta=\frac{-\sqrt{3}}{2} \text{ and } \sin\theta=\frac{1}{2}$$

$$\therefore \theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6} \qquad \left[\text{As } \theta \text{ lies in the II quadrant}\right]$$

Thus, the modulus and argument of the complex number $-\sqrt{3}+i$ are 2 and $\frac{5\pi}{6}$ respectively.

Question 3:

Convert the given complex number in polar form: 1 – *i*

Answer 3:

1 – *i*

Let *r* cos *θ* = 1 and *r* sin *θ* = –1

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + (-1)^2$$

$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1+1$$

$$\Rightarrow r^2 = 2$$

$$\Rightarrow r = \sqrt{2} \qquad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2}\cos\theta = 1 \text{ and } \sqrt{2}\sin\theta = -1$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \text{ and } \sin\theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \theta = -\frac{\pi}{4} \qquad [\text{As } \theta \text{ lies in the IV quadrant}]$$

$$\therefore 1 - i = r\cos\theta + i\,r\sin\theta = \sqrt{2}\cos\left(-\frac{\pi}{4}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{4}\right) = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

This is the required polar form.

Question 4:

Convert the given complex number in polar form: – 1 + *i*

Answer 4:

– 1 + *i*

Let $r\cos\theta = -1$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + 1^2$$

$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1+1$$

$$\Rightarrow r^2 = 2$$

$$\Rightarrow r = \sqrt{2} \qquad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = 1$$

$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad [\text{As } \theta \text{ lies in the II quadrant}]$$

It can be written,

$$\therefore -1+i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

This is the required polar form.

Question 5:

Convert the given complex number in polar form: – 1 – *i*

Answer 5:

– 1 – *i*

Let $r\cos\theta = -1$ and $r\sin\theta = -1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + (-1)^2$$

$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 1+1$$

$$\Rightarrow r^2 = 2$$

$$\Rightarrow r = \sqrt{2} \qquad \left[\text{Conventionally, } r > 0\right]$$

$$\therefore \sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = -1$$

$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \theta = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4} \qquad \left[\text{As } \theta \text{ lies in the III quadrant}\right]$$

$$\therefore -1-i = r\cos\theta + ir\sin\theta = \sqrt{2}\cos\frac{-3\pi}{4} + i\sqrt{2}\sin\frac{-3\pi}{4} = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$$

This is the required polar form.

Question 6:

Convert the given complex number in polar form: −3

Answer 6:

−3

Let r cos θ = −3 and r sin θ = 0

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-3)^2$$

$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 9$$

$$\Rightarrow r^2 = 9$$

$$\Rightarrow r = \sqrt{9} = 3 \qquad [\text{Conventionally, } r > 0]$$

$$\therefore 3\cos\theta = -3 \text{ and } 3\sin\theta = 0$$

$$\Rightarrow \cos\theta = -1 \text{ and } \sin\theta = 0$$

$$\therefore \theta = \pi$$

$$\therefore -3 = r\cos\theta + ir\sin\theta = 3\cos\pi + i3\sin\pi = 3\left(\cos\pi + i\sin\pi\right)$$

This is the required polar form.

Question 7:

Convert the given complex number in polar form: $\sqrt{3}+i$

Answer 7:

$\sqrt{3}+i$

Let r cos θ = $\sqrt{3}$ and r sin θ = 1

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = \left(\sqrt{3}\right)^2 + 1^2$$
$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 3 + 1$$
$$\Rightarrow r^2 = 4$$
$$\Rightarrow r = \sqrt{4} = 2 \qquad \left[\text{Conventionally, } r > 0\right]$$
$$\therefore 2\cos\theta = \sqrt{3} \text{ and } 2\sin\theta = 1$$
$$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = \frac{1}{2}$$
$$\therefore \theta = \frac{\pi}{6} \qquad \left[\text{As } \theta \text{ lies in the I quadrant}\right]$$
$$\therefore \sqrt{3} + i = r\cos\theta + ir\sin\theta = 2\cos\frac{\pi}{6} + i\,2\sin\frac{\pi}{6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

This is the required polar form.

Question 8:

Convert the given complex number in polar form: *i*

Answer 8:

i

Let $r\cos\theta = 0$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = 0^2 + 1^2$$
$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 1$$
$$\Rightarrow r^2 = 1$$
$$\Rightarrow r = \sqrt{1} = 1 \qquad \left[\text{Conventionally, } r > 0\right]$$
$$\therefore \cos\theta = 0 \text{ and } \sin\theta = 1$$
$$\therefore \theta = \frac{\pi}{2}$$
$$\therefore i = r\cos\theta + ir\sin\theta = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

This is the required polar form.

Mathematics

(Chapter – 5) (Complex Numbers and Quadratic Equations)

(Class – XI)

Exercise 5.3

Question 1:

Solve the equation $x^2 + 3 = 0$

Answer 1:

The given quadratic equation is $x^2 + 3 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

we obtain $a = 1$, $b = 0$, and $c = 3$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 0^2 - 4 \times 1 \times 3 = -12$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{\pm\sqrt{-12}}{2 \times 1} = \frac{\pm\sqrt{12}\,i}{2} \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{\pm 2\sqrt{3}\,i}{2} = \pm\sqrt{3}\,i$$

Question 2:

Solve the equation $2x^2 + x + 1 = 0$

Answer 2:

The given quadratic equation is $2x^2 + x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

we obtain $a = 2$, $b = 1$, and $c = 1$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 1^2 - 4 \times 2 \times 1 = 1 - 8 = -7$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2} = \frac{-1 \pm \sqrt{7}\,i}{4} \qquad \left[\sqrt{-1} = i\right]$$

Question 3:

Solve the equation $x^2 + 3x + 9 = 0$

Answer 3:

The given quadratic equation is $x^2 + 3x + 9 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

we obtain $a = 1$, $b = 3$, and $c = 9$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2(1)} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3\sqrt{3}\,i}{2} \qquad \left[\sqrt{-1} = i\right]$$

Question 4:

Solve the equation $-x^2 + x - 2 = 0$

Answer 4:

The given quadratic equation is $-x^2 + x - 2 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

we obtain $a = -1$, $b = 1$, and $c = -2$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2) = 1 - 8 = -7$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times (-1)} = \frac{-1 \pm \sqrt{7}\,i}{-2} \qquad \left[\sqrt{-1} = i\right]$$

Question 5:

Solve the equation $x^2 + 3x + 5 = 0$

Answer 5:

The given quadratic equation is $x^2 + 3x + 5 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain $a = 1$, $b = 3$, and $c = 5$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5 = 9 - 20 = -11$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-11}}{2 \times 1} = \frac{-3 \pm \sqrt{11}i}{2} \qquad \left[\sqrt{-1} = i\right]$$

Question 6:

Solve the equation $x^2 - x + 2 = 0$

Answer 6:

The given quadratic equation is $x^2 - x + 2 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

we obtain $a = 1$, $b = -1$, and $c = 2$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2 \times 1} = \frac{1 \pm \sqrt{7}i}{2} \qquad \left[\sqrt{-1} = i\right]$$

Question 7:

Solve the equation $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Answer 7:

The given quadratic equation is $\sqrt{2}x^2 + x + \sqrt{2} = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{2}$, $b = 1$, and $c = \sqrt{2}$

Therefore, the discriminant of the given equation is

D = $b^2 - 4ac = 1^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}} \qquad \left[\sqrt{-1} = i\right]$$

Question 8:

Solve the equation $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Answer 8:

The given quadratic equation is $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{3}, b = -\sqrt{2}\ and\ c = 3\sqrt{3}$

Therefore, the discriminant of the given equation is

D = $b^2 - 4ac = \left(-\sqrt{2}\right)^2 - 4\left(\sqrt{3}\right)\left(3\sqrt{3}\right) = 2 - 36 = -34$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-\left(-\sqrt{2}\right) \pm \sqrt{-34}}{2 \times \sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}} \qquad \left[\sqrt{-1} = i\right]$$

Question 9:

Solve the equation $x^2 + x + \frac{1}{\sqrt{2}} = 0$

Answer 9:

The given quadratic equation is $x^2 + x + \frac{1}{\sqrt{2}} = 0$

This equation can also be written as $\sqrt{2}x^2 + \sqrt{2}x + 1 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{2}, b = \sqrt{2}$ *and* $c = 1$

$\therefore \text{Discriminant } (D) = b^2 - 4ac = \left(\sqrt{2}\right)^2 - 4 \times \left(\sqrt{2}\right) \times 1 = 2 - 4\sqrt{2}$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-\sqrt{2} \pm \sqrt{2 - 4\sqrt{2}}}{2 \times \sqrt{2}} = \frac{-\sqrt{2} \pm \sqrt{2\left(1 - 2\sqrt{2}\right)}}{2\sqrt{2}}$$

$$= \left(\frac{-\sqrt{2} \pm \sqrt{2}\left(\sqrt{2\sqrt{2} - 1}\right)i}{2\sqrt{2}}\right) \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{-1 \pm \left(\sqrt{2\sqrt{2} - 1}\right)i}{2}$$

Question 10:

Solve the equation $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

Answer 10:

The given quadratic equation is $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

This equation can also be written as $\sqrt{2}x^2 + x + \sqrt{2} = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = \sqrt{2}, b = 1 \text{ and } c = \sqrt{2}$

$\therefore \text{Discriminant } (\text{D}) = b^2 - 4ac = 1^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{\text{D}}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} = \frac{-1 \pm \sqrt{7}\,i}{2\sqrt{2}} \qquad \left[\sqrt{-1} = i\right]$$

Mathematics

(Chapter – 5) (Complex Numbers and Quadratic Equations)

(Class – XI)

Miscellaneous Exercise on chapter 5

Question 1:

Evaluate: $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

Answer 1:

$$\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$$
$$=\left[i^{4\times4+2}+\frac{1}{i^{4\times6+1}}\right]^3$$
$$=\left[\left(i^4\right)^4\cdot i^2+\frac{1}{\left(i^4\right)^6\cdot i}\right]^3$$
$$=\left[i^2+\frac{1}{i}\right]^3 \qquad \left[i^4=1\right]$$
$$=\left[-1+\frac{1}{i}\times\frac{i}{i}\right]^3 \qquad \left[i^2=-1\right]$$
$$=\left[-1+\frac{i}{i^2}\right]^3$$
$$=\left[-1-i\right]^3$$
$$=(-1)^3\left[1+i\right]^3$$
$$=-\left[1^3+i^3+3\cdot1\cdot i(1+i)\right]$$
$$=-\left[1+i^3+3i+3i^2\right]$$
$$=-\left[1-i+3i-3\right]$$
$$=-\left[-2+2i\right]$$
$$=2-2i$$

Question 2:

For any two complex numbers z_1 and z_2, prove that

Re (z_1z_2) = Re z_1 Re z_2 – Im z_1 Im z_2

Answer 2:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\therefore z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$

$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$

$$= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 \qquad [i^2 = -1]$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

$$\Rightarrow \text{Re}(z_1z_2) = x_1x_2 - y_1y_2$$

$$\Rightarrow \text{Re}(z_1z_2) = \text{Re}\, z_1 \,\text{Re}\, z_2 - \text{Im}\, z_1 \,\text{Im}\, z_2$$

Hence, proved.

Question 3:

Reduce $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Answer 3:

$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)=\left[\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$$

$$=\left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right]=\left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$$

$$=\left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right]=\frac{33+31i}{28-10i}=\frac{33+31i}{2(14-5i)}$$

$$=\frac{(33+31i)}{2(14-5i)}\times\frac{(14+5i)}{(14+5i)}$$ [On multiplying numerator and denominator by $(14+5i)$]

$$=\frac{462+165i+434i+155i^2}{2\left[(14)^2-(5i)^2\right]}=\frac{307+599i}{2(196-25i^2)}$$

$$=\frac{307+599i}{2(221)}=\frac{307+599i}{442}=\frac{307}{442}+\frac{599i}{442}$$

This is the required standard form.

Question 4:

If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ prove that $\left(x^2+y^2\right)^2 = \frac{a^2+b^2}{c^2+d^2}$

Answer 4:

$$x - iy = \sqrt{\frac{a-ib}{c-id}}$$

$$= \sqrt{\frac{a-ib}{c-id} \times \frac{c+id}{c+id}} \quad \text{[On multiplying numerator and denominator by } (c+id)\text{]}$$

$$= \sqrt{\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}}$$

$$\therefore (x-iy)^2 = \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

$$\Rightarrow x^2 - y^2 - 2ixy = \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

On comparing real and imaginary parts, we obtain

$$x^2 - y^2 = \frac{ac+bd}{c^2+d^2}, \quad -2xy = \frac{ad-bc}{c^2+d^2} \qquad (1)$$

$$\left(x^2+y^2\right)^2 = \left(x^2-y^2\right)^2 + 4x^2y^2$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{ad-bc}{c^2+d^2}\right)^2 \qquad [\text{Using } (1)]$$

$$= \frac{a^2c^2+b^2d^2+2acbd+a^2d^2+b^2c^2-2adbc}{\left(c^2+d^2\right)^2}$$

$$= \frac{a^2c^2+b^2d^2+a^2d^2+b^2c^2}{\left(c^2+d^2\right)^2}$$

$$= \frac{a^2\left(c^2+d^2\right)+b^2\left(c^2+d^2\right)}{\left(c^2+d^2\right)^2}$$

$$= \frac{\left(c^2+d^2\right)\left(a^2+b^2\right)}{\left(c^2+d^2\right)^2}$$

$$= \frac{a^2+b^2}{c^2+d^2}$$

Hence, proved.

Question 5:

Convert the following in the polar form:

(i) $\frac{1+7i}{(2-i)^2}$, **(ii)** $\frac{1+3i}{1-2i}$

Answer 5:

(i) Here, $z=\frac{1+7i}{(2-i)^2}$

$$=\frac{1+7i}{(2-i)^2}=\frac{1+7i}{4+i^2-4i}=\frac{1+7i}{4-1-4i}$$

$$=\frac{1+7i}{3-4i}\times\frac{3+4i}{3+4i}=\frac{3+4i+21i+28i^2}{3^2+4^2}$$

$$=\frac{3+4i+21i-28}{3^2+4^2}=\frac{-25+25i}{25}$$

$$=-1+i$$

Let $r \cos \theta = -1$ and $r \sin \theta = 1$

On squaring and adding, we

obtain $r^2 (\cos^2 \theta + \sin^2 \theta) = 1$

$+ 1$

$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 2$

$\Rightarrow r^2 = 2$ $\quad [\cos^2 \theta + \sin^2 \theta = 1]$

$\Rightarrow r=\sqrt{2}$ $\quad$ [Conventionally, $r>0$]

$\therefore \sqrt{2}\cos\theta=-1$ and $\sqrt{2}\sin\theta=1$

$\Rightarrow \cos\theta=\frac{-1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$

$\therefore \theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$ $\quad$ [As θ lies in II quadrant]

$\therefore z = r \cos \theta + i\, r \sin \theta$

$$=\sqrt{2}\cos\frac{3\pi}{4}+i\sqrt{2}\sin\frac{3\pi}{4}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

This is the required polar form.

(ii) Here, $z=\frac{1+3i}{1-2i}$

$$=\frac{1+3i}{1-2i}\times\frac{1+2i}{1+2i}$$

$$=\frac{1+2i+3i-6}{1+4}$$

$$=\frac{-5+5i}{5}=-1+i$$

Let $r\cos\theta = -1$ and $r\sin\theta$

$= 1$ On squaring and adding,

we obtain $r^2(\cos^2\theta + \sin^2\theta)$

$= 1 + 1$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$

$\Rightarrow r^2 = 2$ $[\cos^2\theta + \sin^2\theta = 1]$

$\Rightarrow r=\sqrt{2}$ [Conventionally, $r>0$]

$\therefore \sqrt{2}\cos\theta=-1$ and $\sqrt{2}\sin\theta=1$

$\Rightarrow \cos\theta=\frac{-1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$

$\therefore \theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$ [As θ lies in II quadrant]

$\therefore z = r\cos\theta + i\,r\sin\theta$

$$=\sqrt{2}\cos\frac{3\pi}{4}+i\sqrt{2}\sin\frac{3\pi}{4}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

This is the required polar form.

Question 6:

Solve the equation $3x^2 - 4x + \frac{20}{3} = 0$

Answer 6:

The given quadratic equation is $3x^2 - 4x + \frac{20}{3} = 0$

This equation can also be written as $9x^2 - 12x + 20 = 0$

On comparing this equation with $ax^2 + bx + c = 0$,

we obtain $a = 9$, $b = -12$, and $c = 20$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-12) \pm \sqrt{-576}}{2 \times 9} = \frac{12 \pm \sqrt{576}\, i}{18} \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{12 \pm 24i}{18} = \frac{6(2 \pm 4i)}{18} = \frac{2 \pm 4i}{3} = \frac{2}{3} \pm \frac{4}{3} i$$

Question 7:

Solve the equation $x^2 - 2x + \frac{3}{2} = 0$

Answer 7:

The given quadratic equation is $x^2 - 2x + \frac{3}{2} = 0$

This equation can also be written as $2x^2 - 4x + 3 = 0$

On comparing this equation with $ax^2 + bx + c = 0$,

we obtain $a = 2$, $b = -4$, and $c = 3$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-4)^2 - 4 \times 2 \times 3 = 16 - 24 = -8$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-4) \pm \sqrt{-8}}{2 \times 2} = \frac{4 \pm 2\sqrt{2}\,i}{4} \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{2 \pm \sqrt{2}\,i}{2} = 1 \pm \frac{\sqrt{2}}{2} i$$

Question 8:

Solve the equation $27x^2 - 10x + 1 = 0$

Answer 8:

The given quadratic equation is $27x^2 - 10x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

we obtain $a = 27$, $b = -10$, and $c = 1$

Therefore, the discriminant of the given equation is

$D = b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{10 \pm 2\sqrt{2}\,i}{54} \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{5 \pm \sqrt{2}\,i}{27} = \frac{5}{27} \pm \frac{\sqrt{2}}{27} i$$

Question 9:

Solve the equation $21x^2 - 28x + 10 = 0$

Answer 9:

The given quadratic equation is $21x^2 - 28x + 10 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$a = 21$, $b = -28$, and $c = 10$

Therefore, the discriminant of the given equation is

D = $b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10 = 784 - 840 = -56$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{56}\,i}{42}$$

$$= \frac{28 \pm 2\sqrt{14}\,i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42} i = \frac{2}{3} \pm \frac{\sqrt{14}}{21} i$$

Question 10:

If . $z_1 = 2 - i,\ z_2 = 1 + i,$ find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

Answer 10:

$$z_1 = 2 - i,\ z_2 = 1 + i$$

$$\therefore \left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+1}\right|$$

$$= \left|\frac{4}{2-2i}\right| = \left|\frac{4}{2(1-i)}\right|$$

$$= \left|\frac{2}{1-i} \times \frac{1+i}{1+i}\right| = \left|\frac{2(1+i)}{1^2 - i^2}\right|$$

$$= \left|\frac{2(1+i)}{1+1}\right| \qquad \left[i^2 = -1\right]$$

$$= \left|\frac{2(1+i)}{2}\right|$$

$$= |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus, the value of $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$ is $\sqrt{2}$.

Question 11:

If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x+1)^2}$

Answer 11:

$$a+ib=\frac{(x+i)^2}{2x^2+1}$$
$$=\frac{x^2+i^2+2xi}{2x^2+1}$$
$$=\frac{x^2-1+i2x}{2x^2+1}$$
$$=\frac{x^2-1}{2x^2+1}+i\left(\frac{2x}{2x^2+1}\right)$$

On comparing real and imaginary parts, we obtain

$$a=\frac{x^2-1}{2x^2+1} \text{ and } b=\frac{2x}{2x^2+1}$$
$$\therefore a^2+b^2=\left(\frac{x^2-1}{2x^2+1}\right)^2+\left(\frac{2x}{2x^2+1}\right)^2$$
$$=\frac{x^4+1-2x^2+4x^2}{(2x+1)^2}$$
$$=\frac{x^4+1+2x^2}{(2x^2+1)^2}$$
$$=\frac{(x^2+1)^2}{(2x^2+1)^2}$$
$$\therefore a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$$

Hence, proved.

Question 12:

Let $z_1 = 2 - i,\ z_2 = -2 + i$. Find

(i) $\text{Re}\left(\frac{z_1 z_2}{\overline{z}_1}\right)$, **(ii)** $\text{Im}\left(\frac{1}{z_1 \overline{z}_1}\right)$

Answer 12:

$z_1 = 2 - i,\ z_2 = -2 + i$

(i) $z_1 z_2 = (2-i)(-2+i) = -4 + 2i + 2i - i^2 = -4 + 4i - (-1) = -3 + 4i$

$\overline{z}_1 = 2 + i$

$\therefore \frac{z_1 z_2}{\overline{z}_1} = \frac{-3+4i}{2+i}$

On multiplying numerator and denominator by (2 – *i*), we obtain

$$\frac{z_1 z_2}{\overline{z}_1} = \frac{(-3+4i)(2-i)}{(2+i)(2-i)} = \frac{-6+3i+8i-4i^2}{2^2+1^2} = \frac{-6+11i-4(-1)}{2^2+1^2}$$

$$= \frac{-2+11i}{5} = \frac{-2}{5} + \frac{11}{5}i$$

On comparing real parts, we obtain

$$\text{Re}\left(\frac{z_1 z_2}{\overline{z}_1}\right) = \frac{-2}{5}$$

(ii) $\frac{1}{z_1 \overline{z}_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{(2)^2 + (1)^2} = \frac{1}{5}$

On comparing imaginary parts, we obtain

$$\text{Im}\left(\frac{1}{z_1 \overline{z}_1}\right) = 0$$

Question 13:

Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$

Answer 13:

Let $z=\frac{1+2i}{1-3i}$, then

$$z=\frac{1+2i}{1-3i}\times\frac{1+3i}{1+3i}=\frac{1+3i+2i+6i^2}{1^2+3^2}=\frac{1+5i+6(-1)}{1+9}$$

$$=\frac{-5+5i}{10}=\frac{-5}{10}+\frac{5i}{10}=\frac{-1}{2}+\frac{1}{2}i$$

Let $z=r\cos\theta+ir\sin\theta$

i.e., $r\cos\theta=\frac{-1}{2}$ and $r\sin\theta=\frac{1}{2}$

On squaring and adding, we obtain

$$r^2\left(\cos^2\theta+\sin^2\theta\right)=\left(\frac{-1}{2}\right)^2+\left(\frac{1}{2}\right)^2$$

$$\Rightarrow r^2=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\Rightarrow r=\frac{1}{\sqrt{2}} \qquad [\text{Conventionally, } r>0]$$

$$\therefore \frac{1}{\sqrt{2}}\cos\theta=\frac{-1}{2} \text{ and } \frac{1}{\sqrt{2}}\sin\theta=\frac{1}{2}$$

$$\Rightarrow \cos\theta=\frac{-1}{\sqrt{2}} \text{ and } \sin\theta=\frac{1}{\sqrt{2}}$$

$$\therefore \theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4} \qquad [\text{As } \theta \text{ lies in the II quadrant}]$$

Therefore, the modulus and argument of the given complex number are

$\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4}$ respectively.

Question 14:

Find the real numbers *x* and *y* if (*x* – *iy*) (3 + 5*i*) is the conjugate of –6 – 24*i*.

Answer 14:

Let $z=(x-iy)(3+5i)$

$$z=3x+5xi-3yi-5yi^2=3x+5xi-3yi+5y=(3x+5y)+i(5x-3y)$$

$$\therefore \bar{z}=(3x+5y)-i(5x-3y)$$

It is given that, $\bar{z}=-6-24i$

$$\therefore (3x+5y)-i(5x-3y)=-6-24i$$

Equating real and imaginary parts, we obtain

$$3x+5y=-6 \qquad \ldots \text{(i)}$$
$$5x-3y=24 \qquad \ldots \text{(ii)}$$

Multiplying equation (i) by 3 and equation (ii) by 5 and then adding them, we obtain

$$\begin{array}{r} 9x+15y=-18 \\ 25x-15y=120 \\ \hline 34x \quad =102 \end{array}$$

$$\therefore x=\frac{102}{34}=3$$

Putting the value of *x* in equation (i), we obtain

$$3(3)+5y=-6$$
$$\Rightarrow 5y=-6-9=-15$$
$$\Rightarrow y=-3$$

Thus, the values of *x* and *y* are 3 and –3 respectively.

Question 15:

Find the modulus of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$.

Answer 15:

$$\frac{1+i}{1-i}-\frac{1-i}{1+i}=\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$$

$$=\frac{1+i^2+2i-1-i^2+2i}{1^2+1^2}$$

$$=\frac{4i}{2}=2i$$

$$\therefore \left|\frac{1+i}{1-i}-\frac{1-i}{1+i}\right|=|2i|=\sqrt{2^2}=2$$

Question 16:

If $(x + iy)^3 = u + iv$, then show that: $\frac{u}{x}+\frac{v}{y}=4\left(x^2-y^2\right)$

Answer 16:

$$(x+iy)^3=u+iv$$

$$\Rightarrow x^3+(iy)^3+3\cdot x\cdot iy(x+iy)=u+iv$$

$$\Rightarrow x^3+i^3y^3+3x^2yi+3xy^2i^2=u+iv$$

$$\Rightarrow x^3-iy^3+3x^2yi-3xy^2=u+iv$$

$$\Rightarrow \left(x^3-3xy^2\right)+i\left(3x^2y-y^3\right)=u+iv$$

On equating real and imaginary parts, we obtain

$$u = x^3 - 3xy^2,\ v = 3x^2y - y^3$$

$$\therefore \frac{u}{x} + \frac{v}{y} = \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y}$$

$$= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y}$$

$$= x^2 - 3y^2 + 3x^2 - y^2$$

$$= 4x^2 - 4y^2$$

$$= 4(x^2 - y^2)$$

$$\therefore \frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$$

Hence, proved.

Question 17:

If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$

Answer 17:

Let α = *a* + *ib* and β = *x* + *iy*

It is given that, $|\beta| = 1$

$$\therefore \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow x^2 + y^2 = 1 \qquad \ldots \text{(i)}$$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=\left|\frac{(x+iy)-(a+ib)}{1-(a-ib)(x+iy)}\right|$$

$$=\left|\frac{(x-a)+i(y-b)}{1-(ax+aiy-ibx+by)}\right|$$

$$=\left|\frac{(x-a)+i(y-b)}{(1-ax-by)+i(bx-ay)}\right|$$

$$=\frac{|(x-a)+i(y-b)|}{|(1-ax-by)+i(bx-ay)|} \qquad \left[\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right]$$

$$=\frac{\sqrt{(x-a)^2+(y-b)^2}}{\sqrt{(1-ax-by)^2+(bx-ay)^2}}$$

$$=\frac{\sqrt{x^2+a^2-2ax+y^2+b^2-2by}}{\sqrt{1+a^2x^2+b^2y^2-2ax+2abxy-2by+b^2x^2+a^2y^2-2abxy}}$$

$$=\frac{\sqrt{(x^2+y^2)+a^2+b^2-2ax-2by}}{\sqrt{1+a^2(x^2+y^2)+b^2(y^2+x^2)-2ax-2by}}$$

$$=\frac{\sqrt{1+a^2+b^2-2ax-2by}}{\sqrt{1+a^2+b^2-2ax-2by}} \qquad [\text{Using } (1)]$$

$$=1$$

$$\therefore\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=1$$

Question 18:

Find the number of non-zero integral solutions of the equation $|1-i|^x = 2^x$

Answer 18:

$$|1-i|^x = 2^x$$
$$\Rightarrow \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$$
$$\Rightarrow \left(\sqrt{2}\right)^x = 2^x$$
$$\Rightarrow 2^{\frac{x}{2}} = 2^x$$
$$\Rightarrow \frac{x}{2} = x$$
$$\Rightarrow x = 2x$$
$$\Rightarrow 2x - x = 0$$
$$\Rightarrow x = 0$$

Thus, 0 is the only integral solution of the given equation. Therefore, the number of nonzero integral solutions of the given equation is 0.

Question 19:

If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that:

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$.

Answer 19:

$$(a+ib)(c+id)(e+if)(g+ih) = A + iB$$
$$\therefore |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$$
$$\Rightarrow |(a+ib)| \times |(c+id)| \times |(e+if)| \times |(g+ih)| = |A+iB| \qquad [|z_1 z_2| = |z_1||z_2|]$$
$$\Rightarrow \sqrt{a^2+b^2} \times \sqrt{c^2+d^2} \times \sqrt{e^2+f^2} \times \sqrt{g^2+h^2} = \sqrt{A^2+B^2}$$

On squaring both sides, we obtain

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$. Hence proved.

Question 20:

If $\left(\frac{1+i}{1-i}\right)^m = 1$ then find the least positive integral value of *m*.

Answer 20:

$$\left(\frac{1+i}{1-i}\right)^m = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$

$$\Rightarrow \left(\frac{(1+i)^2}{1^2+1^2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1^2+i^2+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1-1+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{2i}{2}\right)^m = 1$$

$$\Rightarrow i^m = 1$$

$\therefore m = 4k$, where k is some integer.

Therefore, the least positive integer is 1.

Thus, the least positive integral value of *m* is 4 (= 4 × 1).