Transcendental and Polynomial Equations

By

Utpal Saikia
Department of Mathematics

Definition

The equation $f(x)=0$ is called *Algebraic* or *Transcendental* according as $f(x)$ is purely a polynomial in x or contains some other functions such as logarithmic, exponential and trigonometric functions etc., *e. g.*,

$$1+\cos x-5x,\ \ x\tan x-\cosh x,\ \ e^{-x}-\sin x, \text{etc.}$$

A polynomial in x of degree n is an expression of the form

$$f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n,$$

where a's are constants and n is a positive integer. The *zeros* or the *roots* of the polynomial $f(x)$ are those values of x for which $f(x)$ is zero. Geometrically, if the graph of $f(x)$ crosses the x-axis at the point $x=a$ then $x=a$ is a root of the equation $f(x)=0$. We conclude that a is a root of the equation $f(x)=0$ if and only if $f(a)=0$.

Methods For Finding The Initial Approximate Value of The Root

To find the real roots of a numerical equation by any method except that of Graeffe, it is necessary first to find an approximate value of the root by any method. The general technique is that we begin with an initial approximate value say x_0 and then find the better approximations $x_1, x_2, \ldots, x_n$ successively by repeating the same method. If at each step of a method the successive approximations approach the root more and more closely, then we say that the method converges.

(i) Graphical Method: Suppose we are to find the roots of the equation $f(x)=0$. Taking a set of rectangular coordinate axes we plot the graph of $y=f(x)$. Then the real roots of the given equation are the abscissae of the points where the graph crosses the x-axis. Obviously, at these points y is zero and so the equation $f(x)=0$ is satisfied. Hence from the graph of the given equation, approximate values for the real roots of the equation can be found. Sometimes when $f(x)$ involves difference of two functions, the approximate values of the real roots of $f(x)=0$ can be found by writing the equation in the form $f_1(x)=f_2(x)$ where $f_1(x)$ and $f_2(x)$ are both functions of x. Then we plot the two equations $y_1=f_1(x)$ and $y_2=f_2(x)$ on the same axes. The real roots of the given equation are the abscissae of the points of intersection of these two curves because at these points $y_1 = y_2$ and so $f_1(x) = f_2(x)$.

(ii) Bisection Method: We know that if a function $f(x)$ is continuous between a and b and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. Let $f(a)$ be negative and $f(b)$ be positive. Also let the approximate value of the root be given by $x_0 = (a+b)/2$. Now if $f(x_0) = 0$ then it ensures that x_0 is a root of the equation $f(x) = 0$. If $f(x_0) \neq 0$ then the root either lies between x_0 and b or between x_0 and a. It depends on whether $f(x_0)$ is negative or positive. Then again we bisect the interval and repeat the process until the root is obtained to the desired accuracy.

(iii) The Method of False Position (Regula-Falsi Method): It is the oldest method for computing the real root of a numerical equation $f(x) = 0$. It is closely similar to the bisection method.

In this method we find a sufficiently small interval (x_1, x_2) in which the root lies. Since the root lies between x_1 and x_2, the graph of $y = f(x)$ must cross the x-axis between $x = x_1$ and $x = x_2$, and hence y_1 and y_2 must be of opposite signs.

This method is based upon the principle that any portion of a smooth curve is practically straight for a short distance. Hence we assume that the graph of $y = f(x)$ is a straight line between the points (x_1, y_1) and (x_2, y_2). The points are on opposite sides of the x-axis.

The x-coordinate of the point of intersection of the straight line joining (x_1, y_1) and (x_2, y_2) and the axis of x gives an approximate value of the desired root. The Fig. 1 represents a magnified view of that part of the graph which lies between (x_1, y_1) and (x_2, y_2).

Now from the similar triangles PAR and PSQ, we have

$$\frac{AR}{AP} = \frac{SQ}{SP} \quad \text{or} \quad \frac{h}{|y_1|} = \frac{x_2 - x_1}{|y_1| + |y_2|} \quad \Rightarrow \quad h = \frac{(x_2 - x_1)|y_1|}{|y_1| + |y_2|}.$$

Hence the approximate value of the desired root

$$= x_1 + AR = x_1 + h = x_1 + \frac{(x_2 - x_1)|y_1|}{|y_1| + |y_2|}.$$

Hence the approximate value of the desired root

$$= x_1 + AR = x_1 + h = x_1 + \frac{(x_2 - x_1)\,|y_1|}{|y_1| + |y_2|}.$$

Let this value be denoted by $x^{(1)}$. Then to find the better approximation, we find $y^{(1)}$ by $y^{(1)} = f(x^{(1)})$. Now either $y^{(1)}$ and y_1 or $y^{(1)}$ and y_2 will be of opposite signs. If $y^{(1)}$ and y_1 are of opposite signs then one root lies in the interval $(x_1, x^{(1)})$. We again apply the method of false position to this interval and get the second approximation.

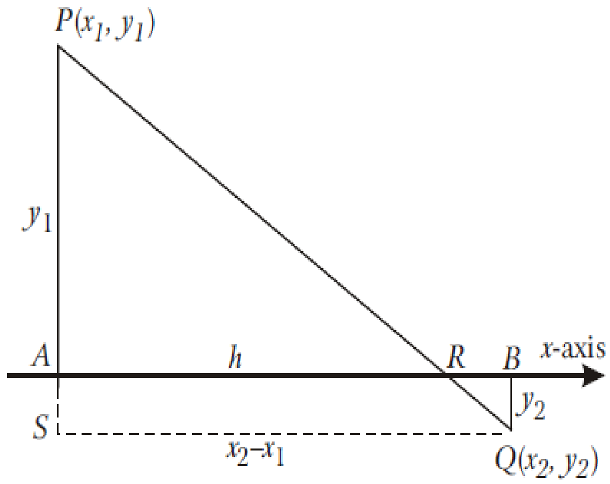


Fig. 1

If $y^{(1)}$ and y_2 are of opposite signs, then second approximation can be obtained by using the method of false position to the interval $(x^{(1)}, x_2)$. Continuing this process we can obtain the root to the desired degree of accuracy.

(iv) The Secant Method: This method is similar to that of Regula-Falsi method except the condition that $f(x_1) \cdot f(x_2) < 0$. In this method the graph of the function $y = f(x)$ in the neighbourhood of the root is approximated by a secant line (chord). Here it is not necessary that the interval at each iteration should contain the root. Let x_1 and x_2 be the limits of interval initially, then the first approximation is given by

$$x_3 = x_2 + \frac{(x_2 - x_1)\, f(x_2)}{f(x_1) - f(x_2)}.$$

The formula for successive approximations in general form is given by

$$x_{n+1} = x_n + \frac{(x_n - x_{n-1}) \cdot f(x_n)}{\{f(x_{n-1}) - f(x_n)\}}, n \geq 2.$$

If at any iteration we have $f(x_n) = f(x_{n-1})$, then the secant method fails. Hence the method does not converge always while the Regula-Falsi method converges surely. But if the secant method converges then it converges more rapidly than the Regula-Falsi method.

(v) Iteration Method: In this method for finding the roots of the equation $f(x) = 0$ it is expressed in the form $x = \phi(x)$.

Let x_0 be an initial approximation to the solution of $x = \phi(x)$. Substituting it in $\phi(x)$ the next approximation x_1 is given by $x_1 = \phi(x_0)$. Again, substituting $x = x_1$ in $\phi(x)$, we get the next approximation as $x_2 = \phi(x_1)$.

Continuing in this way, we get

$$x_n = \phi(x_{n-1}) \quad \text{or} \quad x_{n+1} = \phi(x_n).$$

Thus we get a sequence of successive approximations which may converge to the desired root.

For using this method, the equation $f(x) = 0$ can be put as $x = \phi(x)$ in many different ways.

Let $f(x) = x^2 - x - 1 = 0$. It can be written as

(i) $x = x^2 - 1$,

(ii) $x^2 = x + 1$ or $x = \sqrt{(x+1)}$,

(iii) $x^2 = x + 1$ or $x = 1 + (1/x)$,

(iv) $x = x - (x^2 - x - 1)$ or $x = 2x - x^2 + 1$.

There can be other arrangements also of this equation.

In order to find the root of $f(x) = 0$, *i.e.*, $x = \phi(x)$, we are to find the abscissa of the point of intersection of the line $y = x$ and the curve $y = \phi(x)$. These two may or may not intersect. If these two curves do not intersect, then the equation $f(x) = 0$ has no real root.

Note: The iteration method is convergent conditionally and the condition is that $|\phi'(x)| < 1$ in the neighbourhood of the real root $x = a$.

(vi) Newton-Raphson Method:

Let x_0 denote an approximate value of the desired root of the equation $f(x) = 0$ and let h be the correction which must be applied to x_0 to get the exact value of the root. Then $x_0 + h$ is a root of the equation $f(x) = 0$, so that $f(x_0 + h) = 0$ ...(1)

Expanding $f(x_0 + h)$ by Taylor's theorem, we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0.$$

Now if h is sufficiently small, we may neglect the terms containing second and higher powers of h and get simple relation $f(x_0) + hf'(x_0) = 0$.

This gives $h = -\dfrac{f(x_0)}{f'(x_0)}$, provided $f'(x_0) \neq 0$. The improved value of the root is

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \ldots(2)$$

Successive approximations are given by $x_2, x_3, \ldots, x_{n+1}$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \ldots(3)$$

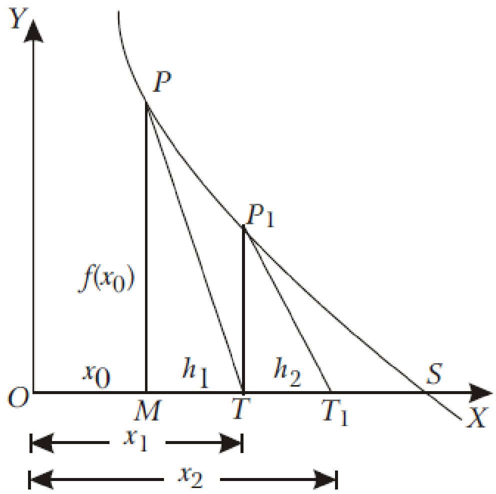
Formula (3) is known as Newton-Raphson formula.

In this method we have assumed that h is a small quantity which is so if the derivative $f'(x)$ is large. In other words the correct value of the root can be obtained more rapidly and with very little labour when the graph is nearly vertical where it crosses the x-axis. If $f'(x)$ is small in the neighbourhood of the root then the value of h is large and by this method the computation of the root will be a slow process or might even fail altogether. Hence this method is not suitable in cases when the graph of $f(x)$ is nearly horizontal where it crosses the x-axis. In such cases the regula-falsi method should be used.

Geometric Significance of the Newton-Raphson Method:

A magnified view of the graph of $y = f(x)$, where it crosses the x-axis is represented in Fig. 2.

Let us draw a tangent at the point P whose x-coordinate is x_0. It intersects the x-axis at some point T. Then we draw a tangent at the point P_1 whose abscissa is OT. Suppose it meets the x-axis at some point T_1 which lies between T and S. Further we draw a tangent at the point P_2 whose abscissa is OT_1. This tangent intersects the x-axis at a point T_2 which lies between T_1 and S. We continue this process. Let the curvature of the graph do not change sign between P and S. Then the points $T, T_1, T_2, \ldots$ will approach the point S as a limit or in other words the intercepts $OT, OT_1, OT_2, \ldots$ will tend to the intercept OS as a limit. But OS denotes the real root of the equation $f(x) = 0$. So $OT, OT_1, OT_2, \ldots$ denote successive approximations to the desired root.



From this figure we derive the fundamental formula. Let $MT = h_1$ and $TT_1 = h_2$, etc.

We have $PM = f(x_0)$, slope at $P = \tan \angle XTP = -\dfrac{f(x_0)}{h_1}$.

Also the slope of the graph at P is $f'(x_0)$.

Thus we get $f'(x_0) = -\dfrac{f(x_0)}{h_1} \Rightarrow h_1 = -\dfrac{f(x_0)}{f'(x_0)}$.

Similarly, we find from the $\Delta P_1 TT_1$ that

$$h_2 = -\frac{f(x_1)}{f'(x_1)}.$$

From this discussion we conclude that in this method we replace the graph of the given function by a tangent at each successive step in the approximation process.

It can be used for solving both algebraic and transcendental equations and it can also be used when the roots are complex.

Newton's iterative formula for obtaining inverse, square root, cube root etc.

1. Inverse. The quantity a^{-1} can be considered as a root of the equation $(1/x) - a = 0$.

Here $f(x) = x^{-1} - a$.

$$\therefore \qquad f'(x) = -1/x^2.$$

Hence by Newton's formula, we get the simple recursion formula

$$x_{n+1} = x_n + \frac{(1/x_n) - a}{1/x_n^2}$$

or

$$x_{n+1} = x_n (2 - ax_n).$$

2. Square root: The quantity $\sqrt{a}$ can be considered as a root of the equation $x^2 - a = 0$. From this we get the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

3. Inverse Square root: The inverse square root of a is the root of the equation $\frac{1}{x^2} - a = 0$. From this we get the iterative formula

$$x_{n+1} = \frac{1}{2} x_n (3 - ax_n^2).$$

4. Formula of p th root and reciprocal p th root: For computing p th root of a we can solve the equation $x^p - a = 0$.

Here $\qquad f(x) = x^p - a. \qquad \therefore \qquad f'(x) = px^{p-1}.$

Hence by Newton's formula, we obtain the recursion formula

$$x_{n+1} = x_n - \frac{(x_n^p - a)}{px_n^{p-1}} = \frac{(p-1)\, x_n^p + a}{px_n^{p-1}}.$$

The reciprocal of pth root of a can be obtained by solving the equation $\frac{1}{x^p} - a = 0$ by Newton's method. We get the iterative formula

$$x_{n+1} = x_n \frac{(p + 1 - ax_n^p)}{p}.$$

-19 Solve

$x^3 - 9x + 1$ by bisection method.

Solⁿ:- $f(x) = x^3 - 9x + 1$

$f(1) = -7 < 0$

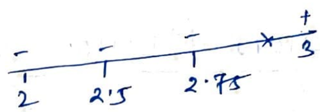
$f(2) = -9 < 0$

$f(3) = 1 > 0$

$\therefore$ Roots lies betⁿ 2 and 3.

$x_1 = \frac{3+2}{2} = 2.5$

$f(2.5) = -5.875 < 0$



Roots lies betⁿ 2.5 and 3.

$x_2 = \frac{2.5+3}{2} = 2.75$

$f(2.75) = -2.96 < 0.$

Roots lies betⁿ 2.75 and 3.

$x_3 = \frac{2.75+3}{2} = 2.875$

$f(2.875) = -1.11 < 0$

Roots lies betⁿ 2.875 and 3.

$x_4 = \frac{2.875+3}{2} = 2.9375.$

$f(2.9375) = -0.0905$

Ex-20 Find real roots of the eqn x^2-5x+3 by bisection method.

Soln

$$f(x) = x^2-5x+3.$$

$$f(1) = -1 < 0$$

$$f(2) = -3 < 0$$

$$f(3) = -3 < 0$$

$$f(4) = -1 < 0$$

$$f(5) = 3 > 0$$

$\therefore$ Roots lies betn 4 and 5.

$$x_1 = \frac{4+5}{2} = 4.5$$

$$f(4.5) = (4.5)^2 - 5\times4.5+3 = .75 > 0.$$

Roots lies betn 4 and 4.5

$$\therefore x_2 = \frac{4+4.5}{2} = 4.25$$

$$f(4.25) = (4.25)^2 - 5\times4.25+3 = -0.1875 < 0$$

Roots lies betn 4.5 and 4.25

$$x_3 = \frac{4.5+4.25}{2} = 4.375$$

$$f(4.375) = (4.375)^2 - 5\times4.375+3 = 0.2656 > 0$$

Roots lies between 4.375 and 4.25

$$x_4 = \frac{4.25+4.375}{2} = 4.3125.$$

~~Ex-20 Find~~

Ex-21 Find real roots of the eqn by Newton Raphson's method $x^3-9x+1=0$

Soln, $f(x) = x^3-9x+1.$

$f(1) = -7 < 0$

$f(2) = -9 < 0$

$f(3) = 1 > 0.$

Roots lies between 2 and 3.

Taking, $x_0 = 3.$

$f'(x) = 3x^2-9.$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{3^3-9\times3+1}{3\times3^2-9}$$

$$= 2.94.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.94 - \frac{(2.94)^3-9\times2.94+1}{3\times(2.94)^2-9}$$

$$= 2.942.$$

$\therefore$ Ans :- $2.94.$

Ex-22 Find a real root of $\sqrt{12}$ by Newton Raphson method.

Soln:-

$x = \sqrt{12}$

$\Rightarrow x^2 = 12 . \Rightarrow x^2 - 12 = 0$

$f(x) = x^2 - 12 = 0$

$\Rightarrow f'(x) = 2x .$

$f(1) = 1-12 = -11 < 0$
$f(2) = 4-12 = -8 < 0$
$f(3) = 9-12 = -3 < 0$
$f(4) = 16-12 = 4 < 0$

Taking, $x_0 = 3.$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 3 - \frac{3^2-12}{2 \times 3} = 3.5 .$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 3.5 - \frac{(3.5)^2-12}{2 \times 3.5} = 3.464 .$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 3.464 - \frac{(3.464)^2 - 12}{2 \times 3.464} = 3.4637$

$= 3.46$

$\therefore$ Ans $= 3.46 .$

Ex-23 Find the real root of $\sqrt[3]{12}$ by Newton Raphson method.

Soln

$x = \sqrt[3]{12}$

$x = \sqrt[3]{12}$
$= (12)^{1/3}$

$\Rightarrow x^3 = 12$

$\Rightarrow x^3 - 12 = 0$

$f(x) = x^3 - 12$

$f'(x) = 3x^2$.

Let, $x_0 = 2$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{2^3 - 12}{3 \cdot 2^2}$$

$$= 7/3 = 2.33$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.33 - \frac{(2.33)^3 - 12}{3 \times (2.33)^2}$$

$$= 2.29.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.29 - \frac{(2.29)^3 - 12}{3 \cdot (2.29)^2}$$

$$= 2.289 \cong 2.29.$$

Ans:- 2.29.

Ex-24 Describe the formula of finding square root of N by Newton Raphson method.

Soln:-

$$x = \sqrt{N}$$

$$\Rightarrow x^2 = N$$

$$\Rightarrow x^2 - N = 0$$

$$f(x) \equiv x^2 - N = 0$$

$$f'(x) = 2x.$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad . \; i = 0, 1, 2 \ldots$$

$$= x_i - \frac{(x_i)^2 - N}{2 \cdot x_i}$$

$$= \frac{2x_i^2 - x_i^2 + N}{2x_i}$$

$$= \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

For cube root (P-128)

$$x_{i+1} = \frac{1}{3}\left[2x_i + \frac{N}{x_i^2}\right]$$

Ex-25 Find the square root of 23 by N.R method.

Soln:

N = 23.

Taking, $x_0 = 4$.

$$x_1 = \frac{1}{2}\left(x_0 + \frac{23}{x_0}\right) = \frac{1}{2}\left(4 + \frac{23}{4}\right) = 4.875$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{23}{x_1}\right) = \frac{1}{2}\left(4.875 + \frac{23}{4.875}\right) = 4.796$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{23}{x_2}\right) = \frac{1}{2}\left(4.796 + \frac{23}{4.796}\right) = 4.7955$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{23}{x_3}\right) = \frac{1}{2}\left(4.7955 + \frac{23}{4.7955}\right) = 4.7958$$

Ex-26 Find the real root of the eqn $x^3 - x - 1 = 0$ by N.R method

Solⁿ:-

$$f(x) = x^3 - x - 1$$

$$f'(x) = 3x^2 - 1$$

$f(1) = -1 < 0$

$f(2) = 5 > 0$.

Roots lies between 1 and 2.

Taking, $x_0 = 1$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{1-1-1}{3.1-1}$$

$$= 1.5.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.5 - \frac{(1.5)^3 - 1.5 - 1}{3.(1.5)^2 - 1}$$

$$= 1.347.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.347 - \frac{(1.347)^3 - 1.347 - 1}{3\times(1.347)^2 - 1}$$

$$= 1.3257$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.325 - \frac{(1.325)^3 - 1.325 - 1}{3\times(1.325)^2 - 1}$$

$$= 1.3249 \cong 1.325.$$

$\therefore$ Ans:- 1.325

Ex-27 Find the Square root of 18. by N.R. method.

Sol^n

$x = \sqrt{18}$

$\Rightarrow x^2 = 18$.

$\Rightarrow x^2 - 18 = 0$.

$f(x) = x^2 - 18$.

$f'(x) = 2x$.

Taking $x_0 = 4$.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 4 - \frac{-2}{8}$

$= 4.25$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 4.25 - \frac{(4.25)^2 - 18}{2 \times 4.25}$

$= 4.2426$.

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 4.24 - \frac{(4.24)^2 - 18}{2 \times 4.24}$

$= 4.2426$

Ans:- 4.24.

Iteration

$y = f(x) = 0$ — (1) be the eqn.

x_0 is initial approximation.

(1) Rewrite of the form.

$$x = \phi(x)$$

$$x_1 = \phi(x_0)$$

$$x_2 = \phi(x_1)$$

$$- - -$$

$$x_{n+1} = \phi(x_n).$$

Ex-28 Find real root of the eqn $2x - \log_{10} x = 7$ by Iteration Method.

Soln:-

$$f(x) = 2x - \log_{10} x - 7$$

$f(1) = 2 \times 1 - \log 1 - 7 = -5 < 0$

$f(2) = 2 \times 2 - \log 2 - 7 = -3.3010 < 0$

$f(3) = 2 \times 3 - \log 3 - 7 = -1.477 < 0.$

$f(4) = 2 \times 4 - \log 4 - 7 = .3979 > 0.$

Roots lies betn 3 and 4.

Take, $x_0 = 3.5$.

Given, Eqn can be expressed -

$$2x = 7 + \log_{10} x.$$

$$\Rightarrow x = \frac{1}{2}(7 + \log_{10} x).$$

$$= \phi(x).$$

$$x_1 = \phi(3.5)$$

$$= \frac{1}{2}(7 + \log_{10} 3.5)$$

$$\approx 3.7720$$

$x_2 = \frac{1}{2}(7 + \log 3.772)$

$= 3.788.$

$x_3 = \frac{1}{2}(7 + \log 3.788)$

$= 3.7892$

$x_4 = \frac{1}{2}(7 + \log 3.7892)$

$= 3.7892.$

Ans:- 3.7892

Ex - 29 Find real root of the eqn $x^2 - 5x + 3$ by Iteration Method.

Sol^n $f(x) = x^2 - 5x + 3.$

$f(1) = 1 - 5\times1 + 3 = -1 < 0$

$f(2) = 2^2 - 5\times2 + 3 = -3 < 0.$

$f(3) = 3^2 - 5\times3 + 3 = -3 < 0.$

$f(4) = 4^2 - 5\times4 + 3 = -1 < 0$

$f(5) = 5^2 - 5\times5 + 3 = 3 > 0.$

Roots lies betn 4 and 5.

Take, $x_0 = 4.5$

Given eqn can be expressed,

$5x = x^2 + 3$

$\Rightarrow x = \frac{1}{5}(x^2 + 3)$

$= \phi(x).$

$x_1 = \phi(x_0) = \phi(4.5)$

$= \frac{1}{5}(4.5^2 + 3)$

$= 4.65.$