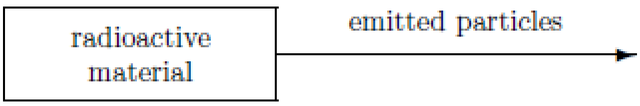
Exponential decay and radioactivity

The process of dating aspects of our environment is essential to the understanding of our history. From the formation of the Earth through the evolution of life and the development of mankind, historians, geologists, archaeologists, palaeontologists and many others use dating procedures to establish theories within their disciplines.

Compartmental diagram

While certain elements are stable, others (or their isotopes) are not, and emit α-particles, β-particles or photons while decaying into isotopes of other elements. Such elements are called radioactive.

The decay, or disintegration, of one nucleus is a random event and so for small numbers of nuclei one might apply probability functions. However, when dealing with large numbers of nuclei we can be reasonably certain that a proportion of the nuclei will decay in any time interval and thus we can model the process as continuous with some fixed rate of decay. We can consider the process in terms of a compartment without input but with output over time, as in Figure



We make the following assumptions and then, based on these, develop a model to describe the process.

- We assume that the amount of an element present is large enough so that we are justified in ignoring random fluctuations.
- We assume the process is continuous in time.
- We assume a fixed rate of decay for an element.
- We assume there is no increase in mass of the body of material.

The first step is to determine an equation describing this disintegration process as

$$\left\{\begin{matrix}\text{rate of change of}\\ \text{radioactive material}\\ \text{at time } t\end{matrix}\right\} = -\left\{\begin{matrix}\text{rate amount of}\\ \text{radioactive}\\ \text{material}\\ \text{decayed}\end{matrix}\right\}. \longrightarrow \quad 1$$

Formulating the differential equation

let $N(t)$ be the number of radioactive nuclei at time t and let Δt be a small change in time. We know that the change in the number of nuclei is proportional to the number of nuclei at the start of the time period. Hence (1) translates to

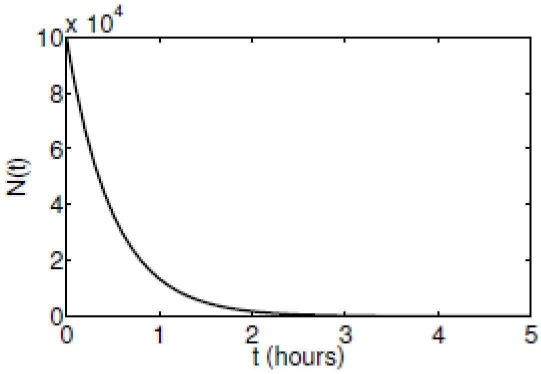
$$\frac{dN}{dt} = -kN$$

where k is a positive constant of proportionality indicating the rate of decay per nucleus in unit time. We assume k to be fixed although it will have a different value for different elements/isotopes.

Here $N(t)$ should be an integer (number of nuclei) while $-kN(t)\Delta t$ may not be. Given a sample of a radioactive element at some initial time, say n_0 nuclei at t_0, we may want to predict the mass of nuclei at some later time t. We require the value of k for the calculations; it is usually found through experimentation. Then, with known k and an initial condition $N(0) = n_0$ we have an initial value problem (IVP)

$$\frac{dN}{dt} = -kN, \qquad N(0) = n_0.$$

We can solve the differential equation numerically, using MATLAB or can also be solve by the technique of separation of variables. The graph of the solution will be as follows



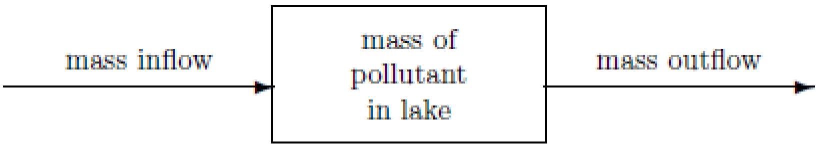
: Numerical solution of the exponential decay differential equation with $n_0 = 10^5$ and $k = 2.0$.

Lake pollution models

Pollution in our lakes and rivers has become a major problem, particularly over the past 50 years. In order to improve this situation in the future, it is necessary to gain a good understanding of the processes involved. Some way of predicting how the situation might improve (or decline) as a result of current management practices is vital. To this end we need to be able to predict how pollutant amounts or concentrations vary over time and under different management strategies.

General compartmental model

This problem can be considered as a compartmental model with a single compartment, the lake, as is illustrated in Figure. Applying the balance law there is an input of polluted water from the river(s) flowing into the lake, or due to a pollution dump into the lake, and an output as water flows from the lake carrying some pollution with it.



This leads us to the word equation, for the mass of pollutant in the lake,

$$\left\{\begin{array}{c}\textit{rate of change}\\ \textit{of mass}\\ \textit{in lake}\end{array}\right\} = \left\{\begin{array}{c}\textit{rate}\\ \textit{mass}\\ \textit{enters lake}\end{array}\right\} - \left\{\begin{array}{c}\textit{rate}\\ \textit{mass}\\ \textit{leaves lake}\end{array}\right\}.$$

Case Study: Lake Burley Griffin

Lake Burley Griffin in Canberra, the capital city of Australia, was created artificially in 1962 for both recreational and aesthetic purposes. In 1974 the public health authorities indicated that pollution standards set down for safe recreational use were being violated and that this was attributable to the sewage works in Queanbeyan upstream (or rather the discharge of untreated sewage into the lake's feeder river).

After extensive measurements of pollution levels taken in the 1970s it was established that, while the sewage plants (of which there are three above the lake) certainly exacerbated the problem, there were significant contributions from rural and urban runoff as well, particularly during summer rainstorms. These contributed to dramatic increases in pollution levels and at times were totally responsible for lifting the concentration levels above the safety limits. As a point of interest, Queanbeyan (where the sewage plants are situated) is in the state of New South Wales (NSW) while the lake is in the Australian Capital Territory, and, although they are a ten-minute drive apart, the safety levels/standards for those who swim in NSW are different from the standards for those who swim in the Capital Territory.

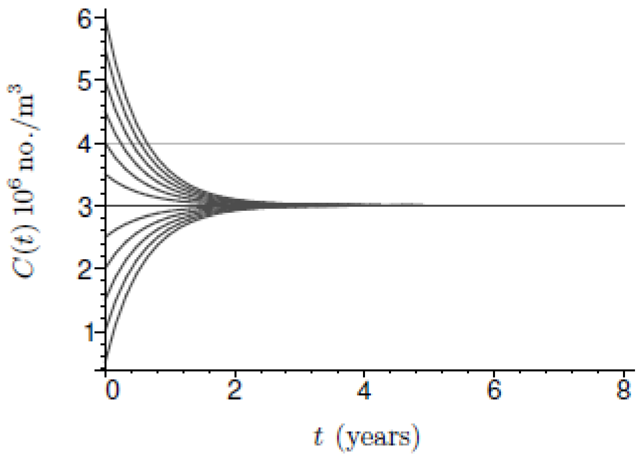
In 1974 the mean concentration of the bacteria faecal coliform count was approximately 10^7 bacteria per m^3 at the point where the river feeds into the lake. The safety threshold or this faecal coliform count in the water is such that for contact recreational sports no more than 10% of total samples over a 30-day period should exceed 4×10^6 bacteria per m^3. Given that the lake was polluted, it is of interest to examine how, if sewage management were improved, the lake would flush out and if and when the pollution levels would drop below the safety threshold. The system can be modelled, very simply, under a few assumptions. Flow (F) into the lake is assumed equal to flow out of the lake, and the volume (V) of the lake will be considered constant and is approximately $28\times10^6 m^3$. Further, the lake can be considered as well mixed in the sense that the pollution concentration throughout will be taken as constant. Under these assumptions a suitable differential equation model for the pollutant concentration is

$$\frac{dC}{dt} = \frac{F}{V}c_{\text{in}} - \frac{F}{V}C,$$

where c_{in} is the concentration of the pollutant entering the lake. With the initial concentration taken as c_0, the solution is

$$C(t) = c_{\text{in}} - (c_{\text{in}} - c_0)e^{-Ft/V}.$$

With only fresh water entering the lake $(c_{in} = 0)$, with a mean monthly flow of $4\times10^6 m^3 month^{-1}$ and with the initial faecal coliform count of 10^7 bacteria per m^3 (as was measured in 1974), the lake will take approximately 6 months for the pollution level to drop below the safety threshold. However, pure water entering the lake is not a very realistic scenario with three sewage plants and much farmland upstream, and so including the entrance of polluted river water into the lake model is essential. From the above solution, as time increases so the concentration of a pollutant in the lake will approach the concentration of the polluted water entering the lake. This level is independent of the initial pollution level in the lake and if $c_0 > c_{in}$ then the level of pollution in the lake decreases monotonically to c_{in}, while if $c_0 < c_{in}$ then the level increases steadily until it reaches c_{in}. Thus, with the faecal coliform entering the lake at a count of 3×10^6 bacteria per m^3 the concentration of the pollutant in the lake will approach this level with time. This is evident in Figure



this model for Lake Burley Griffin is still simplistic in its assumption of a well-mixed body of water. If the concentration decreased from the point of river entry to the point of outflow, then the flushing time could take considerably longer. Further, in most lakes there is a main channel of water flow that flushes regularly, and adjacent to this channel are areas of trapped water that flush less frequently and through a very different process. The process is that of diffusion, which operates at a microscopic level and is extremely gradual. Thus, pockets of the lake may have a much higher (or lower) pollution concentration than others, and these may also be the protected bays where swimming is most likely to take place.