(i) $\sup(S \cup T) = \max\{\sup S, \sup T\}$

(ii) $\inf(S \cup T) = \min\{\inf S, \inf T\}$

- If S and T are two non-empty subset of $\mathbb{R}$ then $\sup(S \cap T) \neq \min\{\sup S, \sup T\}$
- Let S be any non-empty subset of $\mathbb{R}$ having Supremum and let $T = \{\lambda x : x \in S\}$ then T is bounded above $\sup T = \lambda \sup S, \forall \lambda \geq 0$.
- Let S be a non-empty set of $\mathbb{R}$ which has Supremum and let $T = \{\lambda x : x \in S\}$ then T is bounded below $\forall \lambda \leq 0$ and $\inf T = \lambda \sup S$.
- $\sup\{x \in \mathbb{Q} : x < a\} = a$ for each $a \in \mathbb{R}$.
- Finite union and arbitrary intersection of bounded subsets of $\mathbb{R}$ is bounded.
- A subset of bounded set is bounded.
- If S and T are two non empty bounded subsets of $\mathbb{R}$. Then,

(i) $\sup\{x + y : x \in S, y \in T\} = \sup S + \sup T$

(ii) $\inf\{x + y : x \in S, y \in T\} = \inf S + \inf T$

(iii) $\sup\{x - y : x \in S, y \in T\} = \sup S - \inf T$

$\inf\{x - y : x \in S, y \in T\} = \inf S - \sup T$

(iv) $\sup\{xy : x \in S, y \in T\} = \sup S \sup T$

where S and T are bounded subset of positive real numbers.

(v) $\inf\{xy : x \in S, y \in T\} = \inf S \inf T$

where S and T are bounded subset of positive real numbers.

2. SEQUENCES OF REAL NUMBERS

- A map from $\mathbb{N}$ to $\mathbb{R}$ defines a sequence of real numbers when the images are arranged in natural order of natural number i.e., starting with images of n is followed by that of $(n+1)$ i.e., $f : \mathbb{N} \to \mathbb{R}$ is a sequence of real numbers and $\langle f(n) \rangle = \langle f(1), f(2), \ldots \rangle$ we write etc.

  $\langle f(n) \rangle = \langle a_n \rangle \langle b_n \rangle, \langle u_n \rangle, \langle v_n \rangle$ etc. $\forall n \in \mathbb{N}$

  $\langle a_n \rangle = \langle a_1, a_2, a_3, \ldots, \rangle$

- **Range set of a Sequence:** The range set of a sequence is the set consisting of all distinct elements of a sequence and without regard to the position term. Thus the range may be finite or infinite set i.e., the range set of sequence $\langle u_n \rangle$ is given by $\{u_n : n \in \mathbb{N}\}$ or simply by $R(u_n)$.
- **Some Important Example of Sequence:**

(i) $a_n = \begin{cases} 2, & n = 1 \text{ or prime} \\ n, & \text{else} \end{cases}$

$a_n = \begin{cases} 2, & n = 1 \text{ or prime} \\ p, & p \mid n \text{ \& } p \text{ is the least prime} \end{cases}$

(ii) $a_{n+2} = a_n + a_{n+1}$, $a_1 = 1$, $a_2 = 1$

(iii) $a_{n+2} = \frac{1}{2}(a_n + a_{n+1})$

where a_1 & a_2 given.

- **Bounded Sequence:** A sequence $\langle u_n \rangle$ is defined as bounded if its range set is bounded. Hence $\langle u_n \rangle$ is bounded if there exist real numbers k' and k such that $k' \leq u_n \leq k, \forall n \in \mathbb{N}$.

OR

If there exist $k \geq 0$ such that $|u_n| \leq k, \forall n \in \mathbb{N}$.

Note:

(i) Supremum and infimum of the range set is the supremum and infimum of the sequence.

(ii) If the range of the sequence $\langle a_n \rangle$ is finite then there exist $\alpha \in \mathbb{R}$ such that $a_n = \alpha$ for infinitely many values of n however converse need not be true.

- **Monotonic Sequence:** Let $\langle a_n \rangle$ be a sequence of real numbers $\langle a_n \rangle$ is monotonic if either $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$ (Then the sequence is called monotonically increasing or non-decreasing)

OR

$a_n \geq a_{n+1}, \forall n \in \mathbb{N}$ (Then the sequence is called monotonically decreasing or non-increasing).

If $a_n < a_{n+1}, \forall n \in \mathbb{N}$ then $\langle a_n \rangle$ is called stricly increasing or increasing sequence.

If $a_n > a_{n+1}, \forall n \in \mathbb{N}$, then $\langle a_n \rangle$ is called strictly decreasing or decreasing sequence.

- **Limit Point of a Sequence:** A real number p is said to be a limit point or a cluster point of a sequence if every neighbourhood of p contains an infinite number of elements of the given sequence.
  In other words, a real number p is a limit point of a sequence $\langle a_n \rangle$ if for any $\varepsilon > 0$, $a_n \in (p-\varepsilon, p+\varepsilon)$ for infinitely many values of n.
  Remark: A real number p is not a limit point of a sequence $\langle a_n \rangle$, if there exists at least one neighbourhood of p which contains only finite number of elements of $\langle a_n \rangle$.
- **Existence of A Limit Point :**
  Bolzano Weierstrass Theorem: Every bounded sequence has a limit point.
- **Subsequence:** Subsequence of a sequence is defined with the help of sequence of natural number. Consider a map $f : \mathbb{N} \to \mathbb{N}$ defined by $f(k) = n_k$ and let $\langle n_k \rangle$ be a strictly increasing sequence of natural numbers. Then for any sequence $\langle a_n \rangle$, the sequence $\langle a_{n_k} \rangle$ is defined as a subsequence of $\langle a_n \rangle$.
- **Complementary Subsequences:** Let $\langle a_{n_k} \rangle$ and $\langle a_{n_{k'}} \rangle$ are subsequence of $\langle a_n \rangle$ then define $S_1 = \{n_k \mid n_k \in \mathbb{N}\}$ and $S_2 = \{n_{k'} \mid n_{k'} \in \mathbb{N}\}$, then $\langle a_{n_k} \rangle$ and $\langle a_{n_{k'}} \rangle$ are complementary subsequences if :
  (i) $S_1 \cup S_2 = \mathbb{N}$
  (ii) $S_1 \cap S_2 = \phi$
- **Limit of a Sequence :** Let $\langle a_n \rangle$ be a sequence of real number and l be a real number, $(l \in \mathbb{R})$ then we say l is the limit of the sequence $\langle a_n \rangle$ if for any $\varepsilon > 0, \exists\, m \in \mathbb{N}$ such that $|a_n - l| < \varepsilon, \ \forall n \geq m$.
  Symbolically, we write it as $\lim_{n\to\infty} a_n = l$ or $a_n \to l$ as $n \to \infty$.
- **Some Remarks:**
  (i) Limit of a convergent sequence is unique.
  (ii) $\lim_{n\to\infty} |u_n - l| = 0 \Leftrightarrow \lim_{n\to\infty} u_n = l$
  (iii) If $|u_n| \leq |v_n|, \forall n \geq m$ and $\lim_{n\to\infty} |v_n| = 0$ thus $\lim_{n\to\infty} |u_n| = 0$, where $|u_n|$ denotes the absolute value of u_n.
  (iv) If $\lim_{n\to\infty} a_n = a$ and $a_n \geq b, \forall n \geq m, m \in \mathbb{N}$. Then $a \geq b (a, b \in \mathbb{R})$.
  (v) A bounded sequence $\langle a_n \rangle$ is convergent $\Leftrightarrow$ it has unique limit point.
  (vi) Every bounded sequence has a convergent subsequence.
  (vii) If each of the two subsequence $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ of a seqrquence $\langle a_n \rangle$ converges to l. Then $\langle a_n \rangle$ also converges to l. ($\because \langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ are complementary subsequences of $\langle a_n \rangle$).
  (viii) If $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ converges to different limit then $\langle a_n \rangle$ cannot converge.

ADVANCED ANALYSIS OF A SEQUENCE:

- **Limit Superior:** Let $\langle a_n \rangle$ be a sequence of real numbers which is bounded above.
  Define
  $$b_1 = \sup\{a_1, a_2, \ldots, a_{n+1}, \ldots\}$$
  $$b_2 = \sup\{a_2, a_3, a_4, \ldots\}$$
  $$\vdots$$
  $$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$
  Then $\langle b_n \rangle$ is defined which is monotonically decreasing.
  Then limit superior of $\langle a_n \rangle$ is denoted by $\overline{\lim}\, a_n$ or $\limsup_{n\to\infty} a_n$ and is defined as $\inf\{b_1, b_2, b_3, b_4, \ldots\}$.
  If $\langle a_n \rangle$ is not bounded above. Then limit superior of $\langle a_n \rangle$ is defined as
  $$\overline{\lim}\, a_n = \limsup_{n\to\infty} a_n = +\infty$$
- **Limit Inferior:** Let $\langle a_n \rangle$ be a sequence of real numbers which is bounded below:

Define $c_1 = \inf\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$

$c_2 = \inf\{a_2, a_3, \ldots\}$

$\vdots$

$c_n = \inf\{a_n, a_{n+1}, \ldots\}$

Then $c_1 \le c_2 \le c_3 \le \ldots$ i.e., $\langle c_n \rangle$ is monotonically increasing sequence.

Now, limit inferior of $\langle a_n \rangle$ denoted by $\underline{\lim}\, a_n$ or $\liminf_{n\to\infty} \langle a_n \rangle$ is defined to be $\sup\{c_1, c_2, c_3, ..\}$ and if $\langle a_n \rangle$ is not bounded below. Then we define $\underline{\lim}\, a_n = -\infty$.

- **Convergent Sequence:** A sequence $\langle a_n \rangle$ is said to be convergent iff limit superior is equal to the limit inferior and they exist finitely i.e. $\overline{\lim}\, a_n = \underline{\lim}\, a_n = l\,(l \in \mathbb{R})$, then l is called the limit of the sequence $\langle a_n \rangle$.
- **Divergent Sequence:** A sequence $\langle a_n \rangle$ is said to be divergent. If $\underline{\lim}\, a_n = \overline{\lim}\, a_n = \infty$ (infinite) or $\underline{\lim}\, a_n = \overline{\lim}\, a_n = -\infty$ (infinite).
- **Oscillatory Sequence:** A sequence $\langle a_n \rangle$ is said to be oscillatory sequence if $\underline{\lim}\, a_n \ne \overline{\lim}\, a_n$.
- **Finitely Oscillatory Sequence:** A sequence $\langle a_n \rangle$ is said to oscillate finitely if both $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ exist finitely and $\underline{\lim}\, a_n \ne \overline{\lim}\, a_n$.
- **Infinitely Oscillatory Sequence:** A sequence $\langle a_n \rangle$ is said to be oscillate infinitely, if both $\underline{\lim}\, a_n$ and $\overline{\lim}\, a_n$ exist infinitely and $\underline{\lim}\, a_n \ne \overline{\lim}\, a_n$.

Examples : $\langle a_n \rangle = \left((-1)^n \cdot n\right)$

- **The Sequence of Natural Numbers:** A sequence $\langle a_n \rangle$ is said to be sequence of natural number if its range set contains only natural numbers.
- **Results based on sequence of natural numbers:**

  (i) Every sequence of natural number has to be bounded below and has infimum in the range set.

  (ii) If sequence of natural number has limit point p. Then this p has to be natural number & $a_n = p$ for infinite many value of n i.e., there exist subsequence $\langle a_{n_k} \rangle$ which is a constant sequence such that a_{n_k} i.e., a sequence $\langle a_n \rangle$ of $\mathbb{N}$ has a limit point $\Leftrightarrow$ it has a constant subsequence.

  (iii) Sequence of natural number is convergent iff it is eventually constant.

  (iv) If a sequence of natural numbers is not divergent then it has constant subsequence.

- **Cauchy Sequence:** A sequence $\langle a_n \rangle$ is said to be a Cauchy sequence if for any $\varepsilon > 0$, there exists a positive integer m such that $|a_n - a_m| < \varepsilon$, whenever $n \ge m$.

Some Important Theorems:

- **Cauchy's First Theorem on limits:**

  Let $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty}\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) = 0$

  In General: Let $\lim_{n\to\infty} a_n = l$

  Then, $\lim_{n\to\infty}\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) = l$

- **Cauchy's Second Theorem on Limits:**

  If $\langle a_n \rangle$ converges to $l(\ne 0)$ and $a_n > 0$ then $\lim (a_1 a_2 \ldots a_n)^{1/n} = l$

- If all the terms of a sequence $\langle a_n \rangle$ are positive and if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exist. Then $\lim_{n\to\infty} (a_n)^{1/n}$ also exist and the two limits are equal. i.e., $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, provided the later limit exist.

- **Cesaro's Theorem:** Let $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} b_n = b$ where $a, b \in \mathbb{R}$, then

  $$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} = a \cdot b$$

- **Sandwich Theorem:** Let $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle$ are sequence of real number such that

$a_n \le b_n \le c_n, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = l = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = l$.

- If $a_n \ge 0, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = l$, then $l \ge 0$.
- If $\langle a_n\rangle, \langle b_n\rangle$ be two sequence such that $a_n \le b_n, \forall n \in \mathbb{N}$, then $\lim a_n \le \lim b_n$.
- If $\langle a_n\rangle$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, where $l < 1$ then $\lim_{n\to\infty} a_n = 0$.
- Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $\langle S_n\rangle$ & $\langle T_n\rangle$ are two sequences, where $S_n = \max\{a_b, b_n\}$ and $T_n = \min\{a_n, b_n\}$. Then the sequences $\langle S_n\rangle$ and $\langle T_n\rangle$ are convergent and $\lim_{n\to\infty} S_n = \max\{a, b\}$, $\lim_{n\to\infty} T_n = \min\{a, b\}$.
- Let $\langle a_n\rangle$ be a sequence such that $a_n \to a$. Then $a_n^2 \to a^2$. However, the converse may not be true.

  For example, let $\langle a_n\rangle = \langle(-1)^n\rangle$. Then $\langle a_n^2\rangle = \langle 1,1,1,\ldots\rangle$, which converges to 1, but $\langle a_n\rangle$ does not converge to 1.
- Let $\langle a_n\rangle$ be a sequence such that $a_n^2 \to a^2$, then $|a_n| \to |a|$ as $n \to \infty$.

3. SERIES OF REAL NUMBERS

- **Definition:** We know about arithmetic and geometric series etc. A series of n terms is denoted by the expression $u_1 + u_2 + \ldots + u_n$ or $\sum_{i=1}^{n} u_i$. If the series has no last term, then such series is called an infinite series.
- **Infinite Series:** Let $\langle a_n\rangle$ be a sequence of real numbers, $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_{n+1} + \ldots$ then i.e. infinite sum of the members of the sequence, is defined as series of real numbers. It is also denoted by $\sum a_n$.

  For example: $\sum \frac{1}{n}$, here $\langle a_n\rangle = \left\langle \frac{1}{n}\right\rangle$.
- **Sequence of Partial Sums:** Suppose $\sum a_n$ is an infinite series then we define a sequence $\langle S_n\rangle$ as follows:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + \ldots + a_n, \text{ and so on.}$$

  The sequence $\langle S_n\rangle$ is called the sequence of partial sums of the series $\sum a_n$.
- **Convergent Series:** A series $\sum a_n$ is said to be convergent, if the sequence $\langle S_n\rangle$ of partial sums of $\sum a_n$ is convergent and if $\lim_{n\to\infty} S_n = S$ then S is called the sum of the series $\sum a_n$ and then we write it as $S = \sum_{n=1}^{\infty} a_n$.
- **Divergent Series:** The series $\sum a_n$ is said to be divergent, if the sequence $\langle S_n\rangle$ of partial sums of $\sum a_n$ is divergent.
- **Oscillatory Series:** The series $\sum a_n$ is said to oscillatory, if the sequence $\langle S_n\rangle$ of partial sums of $\sum a_n$ oscillates.
- **Necessary condition for convergence of series:** If the series $\sum a_n$ converge, then $\lim_{n\to\infty} a_n = 0$.

Remark:

(i) Converse of the theorem need not be true.

For example: Let $\sum a_n = \sum \frac{1}{n}$, then $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum \frac{1}{n}$ is not convergent.

(ii) If $\lim_{n\to\infty} a_n \ne 0$ then $\sum a_n$ cannot converge.

- **Telescopic Series:**

  Let $\langle a_n\rangle$ be a sequence of real numbers. Define $b_n = a_n - a_{n+1}$ and $c_n = a_{n+1} - a_n$

Then $\sum b_n$ and $\sum c_n$ are called telescopic series.

If $\langle S_n \rangle$ denote the sequence of partial sum of the series $\sum b_n$.

Then $S_n = \sum_{r=1}^{n} b_r$

$= b_1 + b_2 + \dots + b_n$

$= a_1 - a_2 + a_2 - a_3 + \dots + a_n - a_{n+1}$

$= a_1 - a_{n+1}$

Similarly if $\langle t_n \rangle$ denote the sequence of partial sum of the series $\sum c_n$. Then $t_n = a_{n+1} - a_n$.

Thus $\langle S_n \rangle$ and $\langle t_n \rangle$ are convergent iff $\langle a_n \rangle$ is convergent.

Thus $\sum b_n$ and $\sum c_n$ are convergent iff $\langle a_n \rangle$ is convergent.

- **Cauchy's General Principal of Convergence**

A necessary and sufficient condition for a series $\sum a_n$ to converge is that for each $\varepsilon > 0$, there exists a positive integer m, such that, $|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon$ for all $n \geq m$.

Pringsheim's Theorem: If a series $\sum u_n$ of positive monotonic decreasing terms converges then not only $u_n \to 0$ but also $nu_n \to 0$ as $n \to \infty$.

- **Series of Positive Real Numbers:** Series with positive terms are the simplest and the most important type of series one comes across. The simplicity arises mainly from the fact that the sequence of its partial sums is monotonically increasing.

Remark:

(i) A positive term series converges iff the sequence of its partial sum is bounded above.

(ii) The sequence of partial sum of a series with negative terms can be shown to be monotonic decreasing and hence a series with negative terms converges iff the sequence of its partial sum is bounded below.

(iii) It may similarly be seen that a series of negative terms can either converge or diverge to $-\infty$.

(iv) A series $\sum_{n=1}^{\infty} u_n$ whose terms are not necessarily positive may fail to be convergent even if the sequence $\langle S_n \rangle$ is bounded above.

For example: Consider $u_n = (-1)^n$ so that

we have $S_n = \begin{cases} -1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$

Test for Convergence of Positive Terms Series

- **First Comparison Test:** If $\sum u_n$ and $\sum v_n$ are positive terms series, $k > 0$ and $\exists\, m \in \mathbb{N}$ such that $u_n \leq k v_n, \forall n \geq m$:

  (i) $\sum v_n$ converges $\Rightarrow \sum u_n$ converges.

  (ii) $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges.

- **Second Comparison Test:** If $\sum u_n$ and $\sum v_n$ are two positive term series such that

  $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}, \forall n \geq m$ then,

  (i) $\sum v_n$ converges $\Rightarrow \sum u_n$ converges.

  (ii) $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges.

- ***p*-Series Test:** The series $\sum \frac{1}{n^p}$ is convergent iff $p > 1$.

- **Limit form Comparison Test:** Let $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, ($l$ is finite and non-zero). Then $\sum u_n$ and $\sum v_n$ converge or diverge together.

  Remark: If $l = 0$ or $l = \infty$, then the conclusion of the above test may not hold good.

- If $\sum a_n$ is a convergent series of positive terms then $\sum \frac{a_n}{1 + a_n}$ is convergent.

- **Cauchy's nth Root Test :** Let $\sum a_n$ be a positive term series such that $\lim_{n\to\infty} (a_n)^{1/n} = l$.

Then

(i) $\sum a_n$ converges if $l<1$

(ii) $\sum a_n$ diverges if $l>1$

(iii) Test fails if $l=1$

- **Cauchy's Integral Test:** If $u(x)$ is a non-negative, monotonically decreasing and integrable function such that $u(n)=u_n$, $\forall n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} u_n$ is convergent if and only if $\int_1^{\infty} u(x)\,dx$ is convergent.

For example : The series $\sum_{n=1}^{\infty} \frac{1}{n^p} (p>0)$ is convergent if $p>1$ and divergent if $p \le 1$.

For example: The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p>1$ and divergent if $0<p\le 1$.

- **Alternating Series:** A series of the form $u_1 - u_2 + u_3 + \ldots$ where $u_n > 0, \forall n \in \mathbb{N}$ is called an alternating series and is denoted by $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$.

For example: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

- **Leibnitz's Test for Alternating Series:**

If an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ satisfies

(i) $u_{n+1} \le u_n, \forall n$

(ii) $\lim_{n\to\infty} u_n = 0$

Then, the series $\sum (-1)^{n-1} u_n$ converges.

Remark: The alternating series $\sum (-1)^{n-1} u_n$ will not be convergent if either $u_{n+1} \not< u_n, \forall n$ or $\lim_{n\to\infty} u_n \ne 0$.

- **Absolute Convergence:** If $\sum a_n$ is a series of real numbers such that $\sum |a_n|$ is a convergent series. Then $\sum a_n$ is called an absolutely convergent series.

For example:

(i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolute convergent series.

(ii) *Result*: Let $\sum u_n$ be absolutely convergent series. Then $\sum u_n$ is also convergent. The converse of this is not true i.e., a convergent series may not be absolutely convergent.

(iii) *Example*: Consider the series

$$\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

- **Conditional Convergence:** A series $\sum u_n$ is said to be conditionally convergent, if

(i) $\sum u_n$ is convergent, and

(ii) $\sum u_n$ is not absolutely convergent.

For example: Test for convergence and absolute convergence the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots \text{ for } p>0.$$

4. FUNCTION & THEIR PROPERTIES

- **Function:** Let $A, B \subset \mathbb{R}$ be two subsets of $\mathbb{R}$.

A function from A to B denoted by $f: A \to B$ is a rule which assign every element of A to unique element of B. A is defined as a domain of f & B is defined as co-domain of f and $f(A)$ is defined as range of f, where

$f(A) = \{f(x) \in B : x \in A\}$.

- **Equality of two functions:** Two functions f_1 and f_2 are said to be equal if and only if

(i) f_1 and f_2 have same domain D (say)

(ii) $f_1(x) = f_2(x), \forall x \in D$

- **Composition of functions:** Let $f: X \to Y$ and $g: Y \to Z$ be two functions such that $f(x) = y$ and $g(y) = z$, where $x \in X$, $y \in Y$ and $z \in Z$. Then the function $h: X \to Z$ such that $h(x) = z = g(y) = g(f(x)), \forall x \in X$ is known as the composition of f and g and is denoted by gof.

- **Inverse function:** Let $f: X \to Y$ be a one-one onto function. Then the function $g: Y \to X$ which associates to each element $y \in Y$ the unique element $x \in X$ such that $f(x) = y$ is known as inverse function of f. The inverse function g of f is denoted by f^{-1}. Then, we have $f^{-1}: Y \to X$ such that $f^{-1}(y) = x$, where $f(x) = y$.

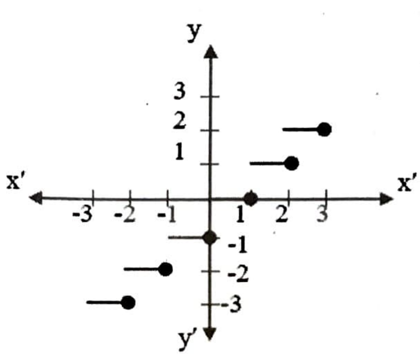
- **Domain of Definition:** Let $y = f(x)$ be a rule, $S \subseteq \mathbb{R}$ on which f becomes real valued function i.e., if S be a subset of $\mathbb{R}$ & $f: S \to \mathbb{R}$ be real valued function then S is called domain of definition. We sometimes denote domain of definition as '*Dod*'.

  For example:

  $f(x) = \sin x,\ x \neq 1,\ Dod = \mathbb{R} \sim \{1\}$

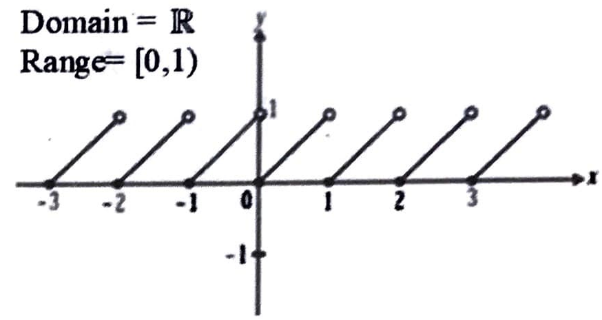
Some well known functions

- **The Greatest Integer Function:** Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x] = n$, $\forall\, n \le x < n+1, n \in \mathbb{N}$, $[x]$ indicates the integral part of x which is nearest and smallest integer to x. It is also known as floor of x. Domain of f is the set of real number and range of f is the set of integers.



For example: $[2.45] = 2, [-2.1] = -3$

- **Fractional Part of Function:** The function $f: \mathbb{R} \to [0,1]$ defined by $f(x) = \{x\} = x - [x]$, $[x]$ is greatest integer function is called fractional part of x

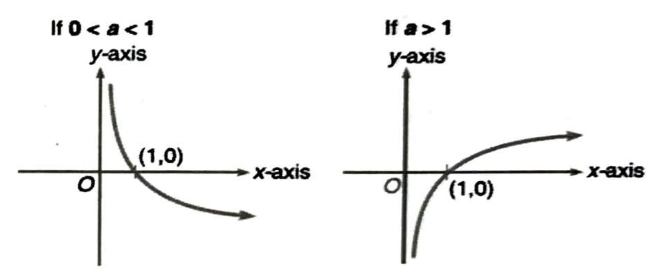


For example: $x = 1.3$ then $\{x\} = 0.3$

- **Least Integer Function:** A function $f: \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \lceil x \rceil =$ the integral part of x which is nearest and greatest integer to x. It is also known as ceiling of x.

  For example: $\lceil 2.3023 \rceil = 3, \lceil -8.0725 \rceil = -8$

- **Logarithmic Function:** Let 'a' be a positive real number, then the function $f: \mathbb{R}^+ \to \mathbb{R}$ defined as $f(x) = \log_a x$ is called the logarithmic function. The range is the set $\mathbb{R}$ of all real numbers.



- **Properties:** Let a, b, c be positive real numbers

  (i) $\log_e(ab) = \log_e a + \log_e b$

  (ii) $\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b$

  (iii) $\log_e a^m = m \log_e a,\ m \in \mathbb{R}$

  (iv) $\log_a a = 1, a \neq 1$

  (v) $\log_{b^m} a = \frac{1}{m} \log_b a$; $b \neq 1$ & $m \in \mathbb{R}$

  (vi) $\log_b a = \frac{1}{\log_a b}; a, b \neq 1$

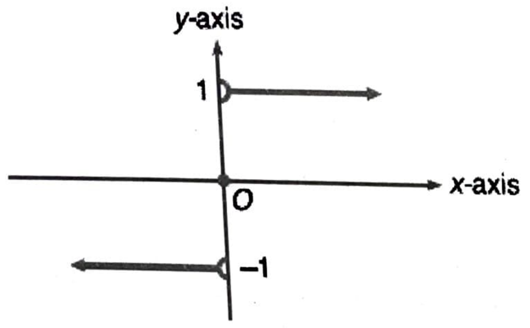
  (vii) $\log_b a = \frac{\log_m a}{\log_m b}; a, b \neq 1$ & $m > 0$

(viii) $a^{\log_a m} = m; m > 0$ & $a \neq 1$

- **Signum Function:** Let $f : \mathbb{R} \to \{-1, 0, 1\}$

defined as $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$

is called signum function.



Classification of Functions

- **Algebraic Function:** A function $y = f(x)$ is said to be an algebraic function and set $S \subset \mathbb{R}$, if it is a root of the equation of the form

$p_0(x)y^n + p_1(x)y^{n-1} + \ldots + p_n(x) = 0$

where $p_i(x)$ are polynomial in $x \in S, 0 \leq i \leq n$

For example: Constant functions are algebraic function even if $f(x) = \pi$.

- **Transcendental Function:** If a function $y = f(x)$ defined on a set $S \subset \mathbb{R}$ is not an algebraic function then it is said to be a transcendental function on S.

For examples: $f(x) = x^{\pi}$

- **Periodic Function:** A function $y = f(x)$ is said to be periodic if $\exists\ l \in \mathbb{R}/\{0\}$ such that $f(x+l) = f(x)$; $\forall\ x \in D.o.d$ and $x + l \in D.o.d.$ and such l is called a period of f.

- **Fundamental Period:** The smallest positive period $l \in \mathbb{R}^+$ for a periodic function f is defined as the fundamental period of f.

For example: $f(x) = \sin x$ is a periodic function with set of periods $\{2n\pi : n \in \mathbb{Z}\}$ fundamental period of $f(x) = \sin x$ is 2π.

Sum or difference of two periodic function may not be periodic.

Sum of two periodic functions (having their fundamental periods) may be periodic but fundamental period may not exist.

Sum or difference of two periodic function may not be periodic.

If f and g are two functions defined on $\mathbb{R}$ such that g is periodic. Then fog is periodic on $\mathbb{R}$.

- **Monotonic Function:**

Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ then

(i) f is said to be monotonically increasing if for $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2); \forall\ x_1, x_2 \in S$

(If strict inequality hold then f is called strictly increasing function)

(ii) f is said to be monotonically decreasing if for $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2); \forall\ x_1, x_2 \in S$

If strict inequality hold then f is called strictly decreasing function.

- **Even Function and Odd Function:** A function $f : S \to \mathbb{R}$ is said to be an even function if $f(-x) = f(x); \forall x \in S$ and f is called an odd function if $f(-x) = -f(x); \forall x \in S$.

Limit of a Function

- **General Principle For Existence of Limit (GPEL):** Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ and $x = a$ is a limit point of S (may or may not be member of S).

Let $f : S \to \mathbb{R}$

We say $l \in \mathbb{R}$ is the limit of $f \Leftrightarrow$ for any $\varepsilon > 0$

$\exists\ \delta > 0$ such that $x_1, x_2 \in \{x : 0 < |x - a| < \delta\}$

$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

- **Second Definition:** Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ and $x = \alpha$ be a limit point of (may or may not belongs to S)

Let $f : S \to \mathbb{R}$.

We say $l \in \mathbb{R}$ is the limit of $f \Leftrightarrow$ for any sequence $\langle a_n \rangle$, such that $\langle a_n \rangle \to \alpha$

$\Rightarrow f(a_n) \to l\ (a_n \in S; \forall n \in \mathbb{N})$

Note: If we can find two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ in S converging to α but $f(a_n)$ and $f(b_n)$ converging to different limit point. Then we say the function has no limit at $x=\alpha$ or limit does not exist at $x=\alpha$.

We can say limit does not exist at $x=\alpha$ if we can find a sequence $\langle a_n \rangle$ in S such that $\langle a_n \rangle \to \alpha$ and $f(a_n)$ does not converge at all.

- **Theorems on Limits**

(i) Uniqueness Theorem: Let $S \subseteq \mathbb{R}$ and f is defined on S.

If $\lim_{x\to a} f(x)$ exist then it is unique.

(ii) If $\lim_{x\to a} f(x) = L$, then f is bounded on some deleted neighbourhood of $x=a$ i.e., if f is unbounded on some neighbourhood of $a \Rightarrow$ limit at a does not exist.

For example: $\lim_{x\to 0} \frac{1}{x}$ does not exist.

(iii) $\lim_{x\to a} f(x)$ and equal to a real number $l \Leftrightarrow$ both left hand limit $\lim_{x\to a^-} f(x)$ and right limit $\lim_{x\to a^+} f(x)$ exist and are equal to l.

- **Sandwich Theorem for Functions:** If function f, g and h are defined on a deleted neighbourhood D of a point a such that

$$f(x) \geq g(x) \geq h(x);\ \forall x \in D$$

and $$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l,$$

then $\lim_{x\to a} g(x)$ exists and equals to l.

- Let f be strictly increasing on $I \subset \mathbb{R}$ then f^{-1} exist and is strictly increasing on $f(I)$.
- If f is monotonic on (a,b), then for each c in (a,b) $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ both exist may not be equal.
- If f is monotonic increasing on (a,b), then for each $c \in (a,b)$

$$\lim_{x\to c^-} f(x) = \sup_{x\in(a,c)} f(x) < f(c) < \inf_{x\in(c,b)} f(x) = \lim_{x\to c^+} f(x)$$

- **Some Important Limits:**

(i) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{-x} = e$

(ii) $\lim_{x\to 0}(1+x)^{1/x} = e$

(iii) $\lim_{x\to\infty}\left(1+\frac{y}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{yx} = e^y$

(iv) $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

(v) $\lim_{x\to 0}\frac{a^x-1}{x} = \log a, \forall a > 0$

(vi) $\lim_{x\to a}\frac{x^p - a^p}{x-a} = pa^{p-1}, \forall\ p \neq 0$ and $a \neq 0$ if $p = 0$

(vii) $\lim_{x\to 0}\frac{\sin x}{x} = 1$

(viii) $\lim_{x\to 0}\cos x = 1$

5. CONTINUITY AND UNIFORM CONTINUITY OF FUNCTIONS

- **Definitions:** Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ be a real valued function. Then we say f is continuous at $\alpha \in S$ if any of the following condition is satisfied:

(i) $\alpha \in S - S'$
(i.e. α is an isolated point of S).

(ii) $\alpha \in S'$ and $\lim_{x\to\alpha} f(x) = f(\alpha)$ (i.e. limit of f exist at $x=\alpha$ and equal to the value of the function).

- **Second Definition of Continuity:** Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ be a function then we say f is continuous at the point $\alpha \in \bar{S} \Leftrightarrow$ for every sequence $\langle x_n \rangle$ in S such that

$$\langle x_n \rangle \to \alpha \Rightarrow f(x_n) \to f(\alpha)$$

Types of Discontinuity :

- **Removable discontinuity (simple discontinuity):** Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ be