Linear Differential Equations With Constant Coefficients

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Some useful results

Let D stand for d/dx ; D^2 for d^2/dx^2; and so on. The symbols D, D^2, etc., are called operators. The index of D indicates the number of times the operation of differentiation must be carried out. For example, D^3x^4 shows that we must differentiate x^4 three times. Thus, $D^3x^4 = 24x$. The following results are valid for such operators.

1. $D^m + D^n = D^n + D^m$
2. $D^mD^n = D^n D^m = D^{m+n}$
3. $D(u + v) = Du + Dv$, where u and v are functions of x.
4. $(D - \alpha)(D - \beta) = (D - \beta)(D - \alpha)$, where α and β are constants.

Negative index of D. D^{-1} is equivalennt to an integration. For example, $D^{-1}x = \int x\, dx = x^2/2$.

Linear differential equations with constant coefficients

A linear differential equation with constant coefficients is that in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together, and the coefficients are all constants.

The general from of the equation is $\dfrac{d^n y}{dx^n} + a_1 \dfrac{d^{n-1}y}{dx^{n-1}} + a_2 \dfrac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = X, \quad \dots (1)$

where X is a function of x only and $a_1, a_2 \dots a_n$ are constants.

Using the symbols D, D^2, ... D^n of Art. 5.1, (1) becomes

$$D^n y + a_1D^{n-1}y + a_2D^{n-2}y + \dots + a_n y = X \quad \text{or} \quad (D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n)y = X \quad \dots (2)$$

or $$f(D)\,y = X \quad \dots (3)$$

where $$f(D) = D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n \quad \dots (4)$$

and $f(D)$ now acts as operator and operates on y to yield X. The forms (2) and (3) are called the symbolic forms of the given equation (1).

Consider the differential equation $$f(D)\,y = 0, \quad \dots (5)$$

obtained on replacing the right hand member of (3) by zero. We will, now, show that if $y_1, y_2 \dots, y_n$ are n linearly independent solutions of (5) then, $c_1y_1 + c_2y_2 + \dots + c_ny_n$ is also a solution of (5); $c_1, c_2 \dots, c_n$ being arbitrary constants.

Since $y_1, y_2 \dots, y_n$ are solutions of (5), $f(D)\,y_1 = 0,\ f(D)\,y_2 = 0,\ \dots f(D)y_n = 0 \quad \dots (6)$

If $c_1, c_2 \dots, c_n$ are an arbitrary constants, we get

$$f(D)(c_1y_1 + c_2y_2 + \dots c_n y_n) = f(D)(c_1y_1) + f(D)(c_2y_2) + \dots + f(D)(c_ny_n)$$
$$= c_1 f(D)\,y_1 + c_2 f(D)\,y_2 + \dots + c_n f(D)\,y_n = c_1.0 + c_2.0 + \dots + c_n.0 = 0, \text{ using (6)}$$

This proves the statement made above.

Since the general solution of a differential equaiton of the n^{th} order contains n arbitrary constants, we conclude that $c_1y_1 + c_2y_2 + \dots + c_n y_n = u$, say

is the general solution of (5).

Thus, $$f(D)\,u = 0. \quad \dots (7)$$

Again, let v be any particular solution of (3) and hence $f(D)\,v = X. \quad \dots (8)$

Now, we have $f(D)(u + v) = f(D)\,u + f(D)\,v = 0 + X$, using (7) and (8)

This shows that $(u + v)$, *i.e.*, $c_1y_1 + c_2y_2 + \dots + c_ny_n + v$ is the general solution of (3), *i.e.*, (1), containing n arbitrary constants $c_1, c_2, \dots, c_n$. The part $c_1y_1 + c_2y_2 + \dots + c_ny_n$, is known as the *ComplementaryFunction* (C.F.) and v, not involving any arbitrary constant, is called the *Particular Integral (P.I.). or particular solution (P.S.)*

Thus, the general solution of (1) is y = C.F. + P.I., where C.F. involves n arbitrary constants and P.I. does not involve any arbitrary constant.

Linear Eqn of 2nd and Higher Order with Constant Co-efficient and Homogeneous.

A differential eqn of the form.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = X$$

where, $P_1, P_2, \ldots P_n$ are constant

is called linear differential eqn of higher order with constant co-efficients.

It can be expressed

$$(D^n + P_1 D^{n-1} + \cdots + P_n) y = X \quad ; \quad D = \frac{d}{dx}.$$

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0 \longrightarrow (1).$$

Let, $y_1, y_2, \ldots y_n$ be the particular soln of (1)

$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is general solution.

Let, $y = e^{mx}$ is a solution of (1).

$$\frac{dy}{dx} = m e^{mx}.$$

$$\frac{d^2 y}{dx^2} = m^2 e^{mx}.$$

...

$$\frac{d^n y}{dx^n} = m^n e^{mx}.$$

Put in (1).

$m^n e^{mx} + P_1 m^{n-1} e^{mx} + \dots + P_n e^{mx} = 0.$

$\Rightarrow e^{mx} (m^n + P_1 m^{n-1} + \dots + P_n) = 0.$

$\therefore m^n + P_1 m^{n-1} + \dots + P_n = 0.$

This is called auxillary eqn.

Case I.

Roots of auxillary eqn real and distinct.

$m = m_1, m_2, \dots m_n$

$\therefore y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$

Ex-1 Solve $(D^2 - 5D + 6)y = 0$

Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y =$

soln:- Auxillary Eqn (A.E)

$m^2 - 5m + 6 = 0.$

$\Rightarrow (m-3)(m-2) = 0$

$\Rightarrow m = 3, 2.$

$y = C_1 e^{3x} + C_2 e^{2x}.$

Solve: $x^2 - 5x + 6 = 0$

$x = ??$

$(D^2 - 5D + 6)y = 0$

where $D = \frac{d}{dx}$

$D^2 =$

C.F $= 0$

Case I

m_1, m_2, m

$y = C_1 e^{m_1 x} + C_2 e$

Ex-2 $(D^3 + 5D^2 + 6D)y = 0$

soln A.E. $m^3 + 5m^2 + 6m = 0.$

$\Rightarrow m(m^2 + 5m + 6) = 0.$

$\Rightarrow m(m+3)(m+2) = 0$

$\Rightarrow m = 0, -3, -2.$

$y = C_1 e^{0x} + C_2 e^{-3x} + C_3 e^{-2x}$

$= C_1 + C_2 e^{-3x} + C_3 e^{-2x}$

Case II   Some roots of auxillary eqn real and equal.

$m = m_4, m_4, m_4, m_4, \ldots m_n.$

$y = (C_1 + C_2 x + C_3 x^2) e^{m_4 x} + C_4 e^{m_4 x} + \ldots + C_n e^{m_n x}$

Ex-3 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$

Soln   Given eqn can be expressed

$(D^2 - 4D + 4)y = 0 \qquad D = \frac{d}{dx}.$

A.E.

$m^2 - 4m + 4 = 0.$

$\Rightarrow (m-2)(m-2) = 0$

$\Rightarrow m = 2, 2.$

$\therefore y = (C_1 + C_2 x) e^{2x}.$

Ex-4   Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$

Soln   Given eqn can be expressed.

$(D^3 - 3D^2 + 3D - 1)y = 0$

A.E. $m^3 - 3m^2 + 3m - 1 = 0.$

$\Rightarrow (m-1)^3 = 0$

$\Rightarrow m = 1, 1, 1$

$y = (C_1 + C_2 x + C_3 x^2) e^x.$

Case III Roots of auxillary eqn imaginary.

$m = \alpha \pm i\beta$, $m_3, m_4, \ldots m_n$.

$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$

Ex-5 Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.

Sm Given eqn can be expressed

$(D^2 + D + 1)y = 0$.

A.E.

$m^2 + m + 1 = 0$.

$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2}$.

$= \frac{-1 \pm i\sqrt{3}}{2}$

$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$y = e^{-\frac{1}{2}x}\left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right)$.

Ex-6 Solve. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0$.

Sm Given, eqn can be expressed.

$(D^2 + D + 2)y = 0$.

A.E.

$m^2 + m + 2 = 0$.

$\Rightarrow m = \frac{-1 \pm \sqrt{1-8}}{2}$

$= -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$

$y = e^{-\frac{1}{2}x}\left(C_1 \cos \frac{\sqrt{7}}{2}x + C_2 \sin \frac{\sqrt{7}}{2}x\right)$

Theory

$$(D^n + P_1 D^{n-1} + \dots + P_n)y = X.$$

$y = (C.F) + (P.I)$.

C.F. = Complementary function.

= Soln when R.H.S is zero.

P.I = Particular integral.

$$= \frac{1}{F(D)} X$$

Case (I)

$$P.I = \frac{1}{F(D)} e^{ax}$$

Than Put $D = a$.

$$= \frac{1}{F(a)} e^{ax}, \quad F(a) \neq 0.$$

Ex-7 Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{4x}$.

Soln Given eqn can be expressed.

$$(D^2 - 3D + 2)y = 2e^{4x}.$$

A.E.

$$m^2 - 3m + 2 = 0.$$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 2, 1.$

$C.F = c_1 e^{2x} + c_2 e^{x}$.

$$P.I. = \frac{1}{D^2 - 3D + 2} 2e^{4x}$$

$$= 2 \frac{1}{4^2 - 3.4 + 2} e^{4x}$$

$$= \frac{2}{6} e^{4x}$$

$$= \frac{1}{3} e^{4x}.$$

$y = C.F + P.I$

$= C_1 e^{2x} + C_2 e^{x} + \frac{1}{3} e^{4x}.$ //

Ex-8 Solve. $(D^2+2D+2)y = 3e^{-x}.$

A.E.

$m^2+2m+2 = 0$

$\Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2}$

$= \frac{-2 \pm 2i}{2}$

$= -1 \pm i$

$C.F = e^{-1x}(C_1 \cos x + C_2 \sin x).$

$P.I = \frac{1}{D^2+2D+2} 3e^{-x}.$

$= 3 \frac{1}{(-1)^2+2(-1)+2} e^{-x}.$

$= 3e^{-x}.$

$y = C.F + P.I.$

$= e^{-x}(C_1 \cos x + C_2 \sin x) + 3e^{-x}.$ //

Case (II) $P.I = \frac{1}{F(D)} e^{ax}.$ $F(a) = 0.$

I[st] Case fail.

Then,

$= x \frac{1}{F'(D)} e^{ax}.$

Ex-9 Solve. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{2x}$.

Given, eqn can be expressed.

$(D^2 - 6D + 8)y = e^{2x}$.

A.E.

$m^2 - 6m + 8 = 0$.

$\Rightarrow (m-4)(m-2) = 0$.

$\Rightarrow m = 4, 2$.

$C.F = C_1 e^{4x} + C_2 e^{2x}$.

$P.I = \frac{1}{D^2 - 6D + 8} e^{2x}$.

$= x \frac{1}{2D - 6} e^{2x}$.

$= x \frac{1}{4-6} e^{2x}$.

$= \frac{x}{-2} e^{2x}$.

$D = \frac{d}{dx}$

$y = C.F + P.I$.

$= C_1 e^{4x} + C_2 e^{2x} - \frac{x}{2} e^{2x}$. //

Ex-10 Solve $(D-3)^2 y = 3e^{3x}$.

A.E $\cancel{(m-3)^2 = 3e^{3x}}$

$\cancel{\Rightarrow m = 3, 3}$

C.F $\cancel{=}$

A.E $(m-3)^2 = 0$

$\Rightarrow m = 3, 3$.

$C.F = (C_1 + C_2 x) e^{3x}$.

$$P.I = \frac{1}{(D-3)^2} 3e^{3x}$$

$$= \frac{1}{D^2-6D+9} 3e^{3x}$$

$$= 3x \frac{1}{2D-6} e^{3x}$$

$$= 3x^2 \frac{1}{2} e^{3x}$$

$$y = C.F + P.I.$$

$$= (c_1 + c_2 x)e^{3x} + \frac{3x^2}{2} e^{3x} \;//$$

Case (III)

$$P.I = \frac{1}{F(D)} \sin ax \text{ or } \cos ax.$$

then Put $D^2 = -a^2$; $F(-a^2) \neq 0$.

Ex-11 Solve $(D^2+2D+1)y = 2\sin 3x$.

A.E. $m^2 + 2m + 1 = 0$.

$\Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$

$$C.F = (c_1 + c_2 x) e^{-x}.$$

$$P.I. = \frac{1}{D^2+2D+1} 2\sin 3x.$$

$$= 2 \frac{1}{-3^2+2D+1} \sin 3x \quad (\text{Putting } D^2 = -3^2).$$

$$= 2 \frac{1}{2D-8} \sin 3x.$$

$$P.I = 2 \cdot \frac{2D+8}{(2D-8)(2D+8)} \sin 3x.$$

$D = \frac{d}{dx}$

$\frac{1}{D} = \int$.

$$= 2 \cdot \frac{2D+8}{4D^2-64} \sin 3x.$$

$$= 2 \cdot \frac{2D+8}{\cancel{4(-3)^2}\ 4(-3^2)-64} \sin 3x.$$

$$= 2 \cdot \frac{2D+8}{-100} \sin 3x.$$

$$= -\frac{1}{50}(2D \sin 3x + 8 \sin 3x).$$

$$= -\frac{1}{50}(2 \cos 3x \cdot 3 + 8 \sin 3x).$$

$$= -\frac{1}{25}(3 \cos 3x + 4 \sin 3x)$$

$y = C.F + P.I$

$$= (c_1 + c_2 x) e^{-x} - \frac{1}{25}(3 \cos 3x + 4 \sin 3x).$$

(iii)

Ex-12 Solve $(D^2 - D - 2)y = \sin 2x$.

Soln :- A.E

$$m^2 - m - 2 = 0.$$

$$\Rightarrow (m-2)(m+1) = 0$$

$$\Rightarrow m = 2, -1$$

$$C.F = \cancel{(c_1 + c_2}\ c_1 e^{2x} + c_2 e^{-x}.$$

$$P.I = \frac{1}{D^2 - D - 2} \sin 2x.$$

$$= \frac{1}{-2^2 - D - 2} \sin 2x.$$

$$= -\frac{1}{(D+6)} \sin 2x.$$

$$P.I = \frac{D-6}{-(D+6)(D-6)} \sin 2x$$

$$= \frac{D-6}{-(D^2-36)} \sin 2x.$$

$$= \frac{D-6}{-(-2^2-36)} \sin 2x.$$

$$= \frac{D-6}{40} \sin 2x.$$

$$= \frac{1}{40}(D \sin 2x - 6 \sin 2x)$$

$$= \frac{1}{40}(2 \cos 2x - 6 \sin 2x)$$

$$= \frac{1}{20}(\cos 2x - 3 \sin 2x).$$

$$y = C.F + P.I.$$

$$= c_1 e^{2x} + c_2 e^{-x} + \frac{1}{20}(\cos 2x - 3 \sin 2x).$$

Case (iv)

$$P.I = \frac{1}{F(D)} \sin ax \text{ or } \cos ax.$$

$D^2 = -a^2.$

then, $F(-a^2) = 0.$

$$= x \frac{1}{F'(D)} \sin ax \text{ or } \cos ax.$$

Ex-13 $(D^2+4)y = 2 \sin 2x.$

A.E. $m^2+4 = 0$

$\Rightarrow m = \pm 2i = 0 \pm 2i$

~~C.F = $e^{0x}(c_1$~~ $\quad C.F = e^{0x}(c_1 \cos 2x + c_2 \sin 2x).$

$$= c_1 \cos 2x + c_2 \sin 2x.$$

$$P.I = \frac{1}{D^2+4} \, 2\sin 2x.$$

$$= 2x \frac{1}{2D+0} \sin 2x. \quad = x \frac{1}{D} \sin 2x$$

$$= x \cdot \left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{x\cos 2x}{2}$$

$$y = C.F + P.I.$$

$$= c_1 \cos 2x + c_2 \sin 2x - \frac{x \cos 2x}{2} \;//$$

Ex-14

Solve $(D^2+25)y = e^x + \cos 5x.$

Sol^n

A.E

$$m^2 + 25 = 0.$$

$$\Rightarrow m = \pm 5i$$

$$C.F = e^{0x}(c_1\cos 5x + c_2 \sin 5x).$$

$$= (c_1 \cos 5x + c_2 \sin 5x).$$

$$P.I = \frac{1}{D^2+25}(e^x + \cos 5x).$$

$$= \frac{1}{D^2+25} e^x + \frac{1}{D^2+25} \cos 5x.$$

$$= \frac{1}{1^2+25} e^x + x \frac{1}{2D} \cos 5x.$$

$$= \frac{e^x}{26} + \frac{x}{2} \frac{\sin 5x}{5}$$

$$= \frac{e^x}{26} + \frac{x}{10} \sin 5x.$$

$$y = C.F + P.I.$$

$$= c_1 \cos 5x + c_2 \sin 5x + \frac{1}{26} e^x + \frac{x}{10} \sin 5x.$$

Case (v)

$$P.I = \frac{1}{F(D)} x^m.$$

$$= [F(D)]^{-1} x^m.$$

Ex-14 Solve $(D^2+3D+2)y = x^2$.

Sm A.E.

$$m^2+3m+2=0.$$

$$\Rightarrow (m+2)(m+1)=0.$$

$$\Rightarrow m=-2,-1.$$

$$C.F = C_1 e^{-2x} + C_2 e^{-x}.$$

$$P.I = \frac{1}{D^2+3D+2} x^2.$$

$$= \frac{1}{2\left[1+\frac{D^2+3D}{2}\right]} x^2.$$

$$= \frac{\left[1+\frac{D^2+3D}{2}\right]^{-1}}{2} x^2.$$

$(1-x)^{-1} = 1+x+x^2+\dots$

$(1+x)^{-1} = 1-x+x^2$

$$= \frac{1}{2}\left[1 - \frac{D^2+3D}{2} + \frac{9D^2}{4}\right]x^2.$$

$$= \frac{1}{2}\left[x^2 - \frac{2+\overset{3\cdot 2x}{\cancel{3x}}}{2} + \frac{9\cdot 2}{4}\right]$$

$$= \frac{1}{2}\left[x^2 - \frac{2+6x}{2} + \frac{9}{2}\right]$$

$$y = C.F + P.I.$$

$$= C_1 e^{-2x} + C_2 e^{-x} + \frac{1}{2}\left[x^2 - \frac{2+6x}{2} + 9/2\right]$$

$$= C_1 e^{-2x} + C_2 e^{-x} + \frac{1}{2}\left(x^2 - 1 + 3x + 9/2\right]$$

Ex-17 $(D^2+4D+4)y = x.$

A.E.

$$m^2+4m+4=0.$$

$$\Rightarrow (m+2)^2=0.$$

$$\Rightarrow m=-2,-2.$$

$$C.F = (c_1+c_2x)e^{-2x}.$$

$$P.I = \frac{1}{D^2+4D+4}x.$$

$$= \frac{1}{4\left(1+\frac{4D+D^2}{4}\right)}x.$$

$$= \frac{1}{4}\left(1+\frac{4D+D^2}{4}\right)^{-1}x.$$

$$= \frac{1}{4}\left(1-\frac{4D}{4}\right)x.$$

$$= \frac{1}{4}(1-D)x.$$

$$= \frac{1}{4}(x-1)$$

$$\therefore y = (c_1+c_2x)e^{-2x} + \frac{1}{4}(x-1).$$ //

Ex-18 Solve. $(D^2-5D+6)y = x+e^{2x}+2.$

A.E.

$$m^2-5m+6=0.$$

$$\Rightarrow (m-3)(m-2)=0$$

$$\Rightarrow m=3,2$$

$$C.F = c_1e^{3x}+c_2e^{2x}.$$

$$P.I = \frac{1}{D^2-5D+6}(x+e^{2x}+2).$$

$$= \frac{1}{D^2-5D+6}(x+e^{2x}+2e^{0x})$$

$$= \frac{1}{D^2-5D+6}x + \frac{1}{D^2-5D+6}e^{2x} + \frac{1}{D^2-5D+6}2e^{0x}$$

$$= \frac{1}{6\left[1+\frac{D^2-5D}{6}\right]}x + x\frac{1}{2D-5}e^{2x} + 2\frac{1}{0-50+6}e^{0x}.$$

$$= \frac{1}{6}\left[1+\frac{D^2-5D}{6}\right]^{-1}x + \frac{x\,e^{2x}}{2\cdot 2-5} + \frac{1}{3}.$$

$$= \frac{1}{6}\left[1+\frac{5D}{6}\right)x - x e^{2x} + 1/3$$

$$= \frac{1}{6}\left(x+\frac{5}{6}\right) - x e^{2x} + 1/3.$$

$y = C.F + P.I.$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{2x} + \frac{1}{6}(x+5/6) - x e^{2x} + 1/3.$$

Case (vi)

$$P.I = \frac{1}{F(D)} X e^{ax}$$

$$= e^{ax}\frac{1}{F(D+a)}X$$

Ex-1

$$(D^2-9)y = e^{3x}\cos x.$$

A.E

$$m^2-9=0.$$

$$\Rightarrow m = 3, -3.$$

$$\therefore C.F = c_1 e^{3x} + c_2 e^{-3x}.$$

$$P.I = \frac{1}{D^2-9} e^{3x} \cos x.$$

$$= e^{3x} \frac{1}{(D+3)^2-9} \cos x.$$

$$= e^{3x} \frac{1}{D^2+6D+9-9} \cos x.$$

$$= e^{3x} \frac{1}{D^2+6D} \cos x.$$

$$= e^{3x} \frac{1}{-1^2+6D} \cos x.$$

$$= e^{3x} \frac{1}{6D-1} \cos x.$$

$$= e^{3x} \frac{6D+1}{(6D-1)(6D+1)} \cos x.$$

$$= e^{3x} \frac{6D+1}{36D^2-1} \cos x.$$

$$= e^{3x} \frac{6D+1}{36(-1)-1} \cos x.$$

$$= e^{3x} \frac{6D+1}{-37} \cos x$$

$$= -\frac{e^{3x}}{37}(6D\cos x + \cos x)$$

$$= -\frac{e^{3x}}{37}(6(-\sin x) + \cos x).$$

$$= \frac{e^{3x}}{37}(6\sin x - \cos x).$$

Ex-10 Solve $(D^2+2D+2)y = xe^{-x}$.

Sol

A.E.

$m^2+2m+2=0$.

$\Rightarrow \quad m = \frac{-2 \pm \sqrt{4-8}}{2}$

$= \frac{-2 \pm 2i}{2}$

$= -1 \pm i \qquad C.F = e^{-x}(c_1 \cos x + c_2 \sin x)$.

$P.I = \frac{1}{D^2+2D+2} xe^{-x}$.

$= e^{-x} \frac{1}{(D-1)^2+2(D-1)+2} x$.

$= e^{-x} \frac{1}{D^2-2D+1+2D-2+2} x$,

$= e^{-x} \frac{1}{D^2+1} x$.

$= e^{-x} (1+D^2)^{-1} x$.

$= e^{-x} (1) x$;

$= xe^{-x}$.

$y = e^{-x}(c_1 \cos x + c_2 \sin x) + xe^{-x}$ //

$$(D^2 - 2D + 1)y = x^2 e^{2x}$$

A.E.

$$m^2 - 2m + 1 = 0.$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1, 1.$$

$$C.F = (c_1 + c_2 x)e^x.$$

$$P.I = \frac{1}{D^2 - 2D + 1} x^2 e^{2x}.$$

$$= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 1} x^2.$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 2D - 4 + 1} x^2.$$

$$= e^{2x} \frac{1}{D^2 + 2D + 1} x^2.$$

$$= e^{2x} [1 + 2D + D^2]^{-1} x^2.$$

$$= e^{2x} (1 - (2D + D^2) + 4D^2) x^2$$

$$= e^{2x} (x^2 - 2 \cdot 2x - 2 + 4 \cdot 2)$$

$$= e^{2x} (x^2 - 4x + 6)$$

$$y = (c_1 + c_2 x)e^x + e^{2x}(x^2 - 4x + 6). \quad //$$

22 $(D^2+2D+2)y = e^{3x}\sin 2x.$

A.E

$$m^2+2m+2=0$$

$$\Rightarrow \quad m=-1\pm i$$

$$C.F = e^{-x}(C_1\cos x + C_2\sin x).$$

$$P.I = \frac{1}{D^2+2D+2} e^{3x}\sin 2x.$$

$$= e^{3x}\frac{1}{(D+3)^2+2(D+3)+2}\sin 2x.$$

$$= e^{3x}\frac{1}{D^2+6D+9+2D+6+2}\sin 2x.$$

$$= e^{3x}\frac{1}{D^2+8D+17}\sin 2x.$$

$$= e^{3x}\frac{1}{-2^2+8D+17}\sin 2x.$$

$$= e^{3x}\frac{1}{8D+13}\sin 2x.$$

$$= e^{3x}\frac{8D-13}{(8D+13)(8D-13)}\sin 2x.$$

$$= e^{3x}\frac{8D-13}{64D^2-169}\sin 2x.$$

$$= e^{3x}\frac{8D-13}{64(-2^2)-169}\sin 2x.$$

$$= e^{3x}.\frac{8D-13}{-256-169}\sin 2x.$$

$$= -\frac{e^{3x}}{425}(8(2\cos 2x)-13\sin 2x)$$

$$P.I = -\frac{e^{3x}}{425}(16\cos 2x - 13\sin 2x).$$

$$y = e^{-x}(C_1\cos x + C_2\sin x) - \frac{e^{3x}}{425}(16\cos 2x - 13\sin 2x) //$$

(23) $(D^3 - 7D - 6)y = x^2 e^{2x}$.

A.E

$$m^3 - 7m - 6 = 0.$$

$$\Rightarrow (m+1)(m^2 - m - 6) = 0.$$

$$\Rightarrow (m+1)(m-3)(m+2) = 0.$$

$$\Rightarrow m = -1, 3, -2.$$

$$C.F = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{-2x}$$

$$P.I = \frac{1}{D^3 - 7D \cancel{+} 6} x^2 e^{2x}.$$

$$= e^{2x} \frac{1}{(D+2)^3 - 7(D+2) \cancel{+} 6} x^2.$$

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 12D + 8 - 7D - 14 \cancel{+} 6} x^2$$

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 5D - 12} x^2.$$

$$= e^{2x} \frac{1}{-12\left[1 - \frac{D^3 + 6D^2 + 5D}{12}\right]} x^2.$$

$$= e^{2x} \frac{\left[1 - \frac{D^3 + 6D^2 + 5D}{12}\right]^{-1}}{-12} x^2.$$

$$= \frac{e^{2x}}{-12}\left[1 + \frac{6D^2 + 5D}{12} + \frac{25D^2}{144}\right] x^2$$

$$= e^{2x}/_{-12} \left(x^2 + \frac{12 + 10x}{12} + \frac{50}{144}\right)$$

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x} - \frac{e^{2x}}{12}\left[x^2 + \frac{12+10x}{12} + \frac{50}{144}\right]$$

(24) $(D^2+9)y = 6e^{3x} + xe^{-3x}$.

Soln

A.E

$m^2 + 9 = 0$.

$\Rightarrow m = \pm 3i$

$C.F = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$.

$= (c_1 \cos 3x + c_2 \sin 3x)$.

$P.I = \frac{1}{D^2+9} 6e^{3x} + xe^{-3x}$.

$= \cancel{6e^{3x} \frac{1}{(D+3)^2+9} xe^{-3}}$

$= \frac{1}{D^2+9} 6e^{3x} + \frac{1}{D^2+9} xe^{-3x}$.

$= 6 \cdot \frac{1}{3^2+9} e^{3x} + e^{-3x} \frac{1}{(D+3)^2+9} x$.

$= \frac{e^{3x}}{3} + e^{-3x} \frac{1}{D^2 \mp 6D + 9 + 9} x$.

$= \frac{e^{3x}}{3} + e^{-3x} \frac{1}{D^2 - 6D + 18} x$.

$= \frac{e^{3x}}{3} + e^{-3x} \frac{\left(1 + \frac{D^2 - 6D}{18}\right)^{-1}}{18} x$.

$= \frac{e^{3x}}{3} + \frac{e^{-3x}}{18}\left(1 + \frac{6D}{18}\right) x$.

$= \frac{e^{3x}}{3} + \frac{e^{-3x}}{18}\left(x + \frac{1}{3}\right)$

$y = c_1 \cos 3x + c_2 \sin 3x + \frac{e^{3x}}{3} + \frac{e^{-3x}}{18}\left(x + \frac{1}{3}\right)$

Case (VII) $\quad P.I = \frac{1}{F(D)} x\, V \qquad V = \sin x$ or $\cos x$.

$e^{ix} = \cos x + i\sin x$.

5) $\frac{d^2y}{dx^2} + 4y = x\sin x$.

$\Rightarrow (D^2+4)y = x\sin x$.

A.E.

$m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

$C.F = C_1\cos 2x + C_2\sin 2x$.

$P.I = \frac{1}{D^2+4} x\sin x$.

$= I.P$ of $\frac{1}{D^2+4} x e^{ix}$.

$= I.P$ of $e^{ix} \frac{1}{(D+i)^2+4} x$.

$= I.P$ of $e^{ix} \frac{1}{D^2+2Di+3} x$.

$=$ " " $e^{ix} \frac{\left(1+\frac{D^2+2Di}{3}\right)^{-1}}{3} x$.

$=$ " " $e^{ix} \frac{\left(1-\frac{2Di}{3}\right)x}{3}$

$=$ " " $\frac{e^{ix}}{3}\left(x - \frac{2i}{3}\right)$.

$=$ " " $\left(\frac{x}{3} - \frac{2i}{9}\right)(\cos x + i\sin x.)$

$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$.

$y = C_1\cos 2x + C_2\sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$.

(26) $\frac{d^2y}{dx^2} + y = x\cos x.$

$(D^2+1)y = x\cos x.$

A.E.

$m^2+1=0.$

$\Rightarrow \quad m = \pm i$

$C.F = c_1\cos x + c_2\sin x.$

$P.I = \frac{1}{D^2+1} x\cos x.$

$= R.P \text{ of } \frac{1}{D^2+1} x \cancel{\cos x}\, e^{ix}.$

$= R.P \text{ of } e^{ix} \frac{1}{(D+i)^2+1} x.$

$= \text{" " } e^{ix} \frac{1}{D^2+2Di} x.$

$= \text{" " } e^{ix} \frac{1}{2Di\left(1+\frac{D^2}{2Di}\right)} x.$

$= \text{" " } e^{ix} \frac{\left(1+\frac{D}{2i}\right)^{-1} x}{2Di}$

$= \text{" " " } \frac{e^{ix}}{2Di}\left(1-\frac{D}{2i}\right)x.$

$= \text{" " " } \frac{e^{ix}}{2Di}\left(x-\frac{1}{2i}\right)$

$= \text{" " " } \frac{e^{ix}}{2i}\left(\frac{x^2}{2}-\frac{x}{2i}\right).$

$= \text{" " " } \left(\frac{x^2}{4i}+\frac{x}{4}\right)(\cos x + i\sin x)$

~~$= \left(\frac{x^2}{4}\cos x\right.$~~

$= \text{" " " } \left(-\frac{x^2}{4}i+\frac{x}{4}\right)(\cos x + i\sin x)$

$$P.I = -\frac{x^2}{4}\sin x(-1) + \frac{x}{4}\cos x.$$

$$= \frac{x^2}{4}\sin x + \frac{x}{4}\cos x.$$

$$y = c_1\cos x + c_2\sin x + \frac{x^2}{4}\sin x + \frac{x}{4}\cos x.$$

7) $D^2 - 5D + 6 = x\sin 3x.$

Sol. A.E

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-3)(m-2) = 0.$$

$$\Rightarrow m = 3, 2.$$

$$C.F = c_1 e^{3x} + c_2 e^{2x}.$$

$$P.I = \frac{1}{D^2 - 5D + 6} x\sin 3x.$$

$$= I.P \text{ of } \frac{1}{D^2 - 5D + 6} x e^{i3x}.$$

$$= I.P \text{ of } e^{i3x} \frac{1}{(D+3i)^2 - 5(D+3i) + 6} x.$$

$$= \text{"} \quad \text{"} \quad e^{i3x} \frac{1}{D^2 + 6Di - 9 - 5D - 15i + 6} x.$$

$$= \text{"} \quad \text{"} \quad e^{3ix} \frac{1}{D^2 + (6i-5)D - 15i - 3} x.$$

$$= \text{"} \quad \text{"} \quad \frac{e^{3ix}\left(1 - \frac{D^2 + (6i-5)D - 15i}{3}\right)^{-1}}{-3} x.$$

$$= \text{"} \quad \text{"} \quad \frac{e^{3ix}}{-3}\left(1 + \frac{(6i-5)D - 15i}{3} + \frac{225i^2}{9}\right)x.$$

$$= \text{"} \quad \text{"} \quad \frac{e^{3ix}}{-3}\left(x + \frac{6i - 5 - 15xi}{3} + \frac{225}{9}(-1)x\right)$$

$$= \text{"} \quad \text{"} \quad -\frac{1}{3}\left[x\left(1 - \frac{225}{9}\right) + \frac{i(6-15x)}{3} - \frac{5}{3}\right]$$

$(\cos 3x -$

$$P.I = -\frac{1}{3}\left[\left(1-\frac{225}{9}\right)x\sin 3x + \frac{6-15x}{3}\cos 3x - \frac{5}{3}\sin 3x\right]$$

$$y = C.F + P.I$$

$$= C_1 e^{3x} + C_2 e^{2x} - \frac{1}{3}\left[\left(1-\frac{225}{9}\right)x\sin 3x + \frac{6-15x}{3}\cos 3x - \frac{5}{3}\sin 3x\right].$$

Homogeneous Equation

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_n y = X$$

we,

$$x = e^z \qquad z = \log x.$$

$$xDy = x\frac{dy}{dx} = x\frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$= x\frac{dy}{dz}\cdot\frac{1}{x}.$$

$$xDy = D'y \quad , \quad D' = \frac{d}{dz}.$$

$\Rightarrow$ $xD = D'$.

$$x^2 D^2 = D'(D'-1)$$

$$x^3 D^3 = D'(D'-1)(D'-2).$$

— —

28) $(x^2D^2 - 4xD + 6)y = x$. $\longrightarrow$ (1).

Let, $x = e^z$ $\quad z = \log x$.

$xD = D'$

$x^2D^2 = D'(D'-1)$. $\quad D' = \frac{d}{dz}$.

: Put in (1).

$(D'(D'-1) - 4D' + 6)y = e^z$.

$\Rightarrow (D'^2 - 5D' + 6)y = e^z$.

A.E.

$m^2 - 5m + 6 = 0$

$\Rightarrow (m-3)(m-2) = 0$

$\Rightarrow m = 3, 2$.

$C.F = C_1e^{3z} + C_2e^{2z}$.

$P.I = \frac{1}{D'^2 - 5D' + 6} e^z$.

$= \frac{1}{1^2 - 5 + 6} e^z$.

$= \frac{e^z}{2}$

$y = C_1e^{3z} + C_2e^{2z} + \frac{e^z}{2}$

$= C_1x^3 + C_2x^2 + \frac{1}{2}x$.

9

20/11

(29) $(x^2D^2 + xD - 1)y = \sin(\log x) + x\cos(\log x).$

Let, $x = e^z \Rightarrow z = \log x.$

$xD = D'$ $\quad D' = \frac{d}{dz}.$

$x^2D^2 = D'(D'-1)$

Put ~~it~~ in the given eqn-

$(D'(D'-1) + D' - 1)y = \sin z + e^z\cos z.$

$\Rightarrow (D'^2 - 1)y = \sin z + e^z\cos z.$

A.E. $m^2 - 1 = 0$

$\Rightarrow m = 1, -1$

$C.F = c_1e^z + c_2e^{-z}$

$P.I = \frac{1}{D'^2 - 1}\sin z + e^z\cos z.$

$= \frac{1}{D'^2 - 1}\sin z + \frac{1}{D'^2 - 1}e^z\cos z.$

$= \frac{1}{-1^2 - 1}\sin z + e^z\frac{1}{(D'+1)^2 - 1}\cos z.$

$= -\frac{\sin z}{2} + e^z\frac{1}{D'^2 + 2D'}\cos z.$

$= -\frac{\sin z}{2} + e^z\frac{1}{-1^2 + 2D}\cos z$

$= -\frac{\sin z}{2} + e^z\frac{(2D+1)}{4D^2 - 1}\cos z.$

$= -\frac{\sin z}{2} + e^z\frac{(2D+1)}{-5}\cos z.$

$= -\frac{\sin z}{2} + \frac{2\sin z}{5}e^z - \frac{e^z\cos z}{5}$

$$y = c_1 e^{z} + c_2 e^{-z} - \frac{\sin z}{2} + \frac{2e^{z}}{5}\sin z - \frac{e^{z}}{5}\cos z.$$

$$= c_1 x + c_2 \frac{1}{x} - \frac{\sin(\log x)}{2} + \frac{2x}{5}\sin(\log x) - \frac{x}{5}\cos(\log x)$$

(38) $x^2D^2 + 5xD + 4y = x^4.$

ut, $e^{z} = x \quad \log x = z.$

$xD = D' \qquad D' = \frac{d}{dz}.$

$x^2D^2 = D'(D'-1)$

Put in the given eqn

$$(D'(D'-1) + 5D' + 4)y = e^{4z}.$$

$$\Rightarrow (D'^2 + 4D' + 4)y = e^{4z}.$$

$\Rightarrow$ A. E $\quad m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$

$\Rightarrow m = -2, -2$

$\therefore$ C.F $= (c_1 + c_2 z)e^{-2z}.$

$$P.I = \frac{1}{D'^2 + 4D' + 4} e^{4z}.$$

$$= \frac{1}{4^2 + 4\cdot 4 + 4} e^{4z}.$$

$$= \frac{e^{4z}}{36}.$$

$$y = (c_1 + c_2 z)e^{-2z} + \frac{e^{4z}}{36}$$

$$= (c_1 + c_2 \log x)\frac{1}{x^2} + \frac{x^4}{36}$$

(31) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$.

$x^3D^3 + 2x^2D^2 + 2y = 10(x + x^{-1})$.

Let, $e^z = x$ $\quad z = \log x$.

$xD = D'$

$x^2D^2 = D'(D'-1)$

$x^3D^3 = D'(D'-1)(D'-2)$.

Put in the given eqn.

$(D'(D'-1)(D'-2) + 2D'(D'-1) + 2)y = 10(e^z + e^{-z})$

$\Rightarrow (D'^3 - 2D'^2 - D'^2 + 2D' + 2D'^2 - 2D' + 2)y = 10(e^z + e^{-z})$

$\Rightarrow (D'^3 - D'^2 + 2)y = 10(e^z + e^{-z})$

A.E $\quad m^3 - m^2 + 2 = 0$

$\Rightarrow (m+1)(m^2 - 2m + 2) = 0$

$\therefore m + 1 = 0 \quad$ or $\quad m^2 - 2m + 2 = 0$

$\Rightarrow m = 1 \pm i$

$\Rightarrow m = -1$

$C.F = C_1e^{-z} + e^z(C_2 \cos z + C_3 \sin z)$.

$P.I. = \frac{1}{D'^3 - D'^2 + 2} 10(e^z + e^{-z})$.

$= 10\left(\frac{1}{D'^3 - D'^2 + 2} e^z + \frac{1}{D'^3 - D'^2 + 2} e^{-z}\right)$

$= 10\left(\frac{1}{1^3 - 1^2 + 2} e^z + z\frac{1}{3D'^2 - 2D'} e^{-z}\right)$

$= 10\left(\frac{e^z}{2} + \frac{ze^{-z}}{5}\right) = 5e^z + 2ze^{-z}$

$y = C_1 e^{-z} + e^{z}(C_2 \cos z + C_3 \sin z) + 5e^{z} + 2ze^{-z}$.

$= C_1 \frac{1}{x} + x(C_2 \cos(\log x) + C_3 \sin(\log x)) + 5x + 2\log x\left(\frac{1}{x}\right)$

(32) $(x^2D^2 + 4xD + 2)y = e^{x}$.

Let; $x = e^{z}$ $\quad z = \log x$.

$xD = D'$

$x^2D' = D'(D'-1)$

Put it in the given eqn.

$(D'(D'-1) + 4D' + 2)y = e^{e^{z}}$

$\Rightarrow (D'^2 + 3D' + 2)y = e^{e^{z}}$.

A.E

$m^2 + 3m + 2 = 0$.

$\Rightarrow (m+2)(m+1) = 0$

$\Rightarrow m = -2, -1$.

$C.F = C_1 e^{-z} + C_2 e^{-2z}$.

$P.I = \frac{1}{D'^2 + 3D' + 2} e^{e^{z}} = \frac{1}{(D'+1)(D'+2)} e^{e^{z}}$

$= \left(\frac{1}{D'+1} - \frac{1}{(D'+2)}\right) \cdot e^{e^{z}}$

$= \frac{1}{D'+1} e^{e^{z}} - \frac{1}{D'+2} e^{e^{z}}$.

$= u - v \longrightarrow (1)$.

$$U = \frac{1}{D'+1} e^{e^z}$$

$$\Rightarrow (D'+1)u = e^{e^z}$$

$$\Rightarrow \frac{du}{dz} + u = e^{e^z}.$$

$$P = 1, \quad Q = e^{e^z}.$$

$$I.F = e^{\int P\,dz}$$

$$= e^{\int dz}$$

$$= e^z$$

$$u(I.F) = \int I.F\, Q\, dz.$$

$$u(e^z) = \int e^z e^{e^z} dz$$

$e^z = t.$

$dt = e^z dz.$

$$\Rightarrow u e^z = \int e^t dt.$$

$$\Rightarrow u e^z = e^t.$$

$$\Rightarrow u e^z = e^{e^z}$$

$$\Rightarrow u = \frac{e^{e^z}}{e^z}$$

$$V = \frac{1}{D'+2} e^{e^z}$$

$$\Rightarrow (D'+2)V = e^{e^z}$$

$$\Rightarrow \frac{dV}{dz} + 2V = e^{e^z}.$$

$$P = 2, \quad Q = e^{e^z}.$$

~~$I.F = e^{\int P dz}$.~~

$$I.F = e^{\int P dz}.$$

$$= e^{\int 2 dz}$$

$$= e^{2z}$$

$$V(I\cdot F) = \int I\cdot F\, Q\, dz$$

$$V e^{2z} = \int e^{2z} e^{e^z} dz.$$

$$= \cancel{\int e^{2z}} \quad = \int e^z \cdot e^z \cdot e^{e^z} dz.$$

$$= \int t e^t dt$$

$$= t\int e^t dt - \left\{\int \frac{d}{dx} t \int e^t dt\right\} dt.$$

$$= te^t - e^t.$$

$$\Rightarrow \quad V = \frac{e^z e^{e^z} - e^{e^z}}{e^{2z}}$$

$$\therefore \; PI = U - V$$

$$= \frac{e^{e^z}}{e^z} - \frac{e^z e^{e^z} - e^{e^z}}{e^{2z}}$$

$$y = c_1 e^{-z} + c_2 e^{-2z} + \frac{e^{e^z}}{e^z} - \frac{e^z e^{e^z} - e^{e^z}}{e^{2z}}$$

$$= c_1 \frac{1}{x} + c_2/x^2 + \frac{e^x}{x} - \frac{x e^x - e^x}{x^2}$$

(33) $$(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x.$$

$$\left((x+a)^2 D^2 - 4(x+a) D + 6\right) y = x.$$

W.r, $\quad x+a = e^z \qquad z = \log(x+a)$

$$(x+a)D = D'$$

$$(x+a)^2 D^2 = D'(D'-1)$$

Put in given eqn.

$$(D'(D'-1) - 4D' + 6)\, y = e^{z} - a$$

$$\Rightarrow (D'^2 - 5D' + 6)\, y = e^{z} - a$$

A.E

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-3)(m-2) = 0$$

$$\Rightarrow m = 3, 2.$$

$$C.F = c_1 e^{3z} + c_2 e^{2z}$$

$$P.I = \frac{1}{D'^2 - 5D' + 6}\, e^{z} - a.$$

$$= \frac{1}{D'^2 - 5D' + 6}\, e^{z} - \frac{1}{D'^2 - 5D' + 6}\, a e^{0z}$$

$$= \frac{1}{1-5+6}\, e^{z} - \frac{a}{0-0+6}$$

$$= \frac{e^{z}}{2} - \frac{a}{6}.$$

$$y = c_1 e^{3z} + c_2 e^{2z} + \frac{e^{z}}{2} - \frac{a}{6}.$$

$$= c_1 (x+a)^3 + c_2 (x+a)^2 + \frac{(x+a)}{2} - \frac{a}{6}$$

4) $[(3x+2)^2 D^2 + 3(3x+2) D - 36]\, y = 3x^2 + 4x + 1$

let,

$3x+2 = e^z \qquad z = \log(3\cancel{x}+2).$

$(3x+2) D = 3D'$

$(3x+2)^2 D^2 = 9(D')(D'-1).$

Put in given eqn :

$(9D'(D-1) + 3.3D' - 36) y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$

$\Rightarrow (9D'^2 - 36) y = \dfrac{e^{2z} - 4e^z + 4 + 4e^z - 8 + 3}{3}$

$\Rightarrow (9D'^2 - 36) y = \dfrac{e^{2z} - 1}{3}$

A.E

$9m^2 - 36 = 0$

$\Rightarrow m = \pm 2$

$C.F = c_1 e^{2z} + c_2 e^{-2z}.$

$P.I = \dfrac{1}{9D'^2 - 36} \dfrac{e^{2z} - 1}{3}.$

$= \dfrac{1}{3}\left(\dfrac{1}{9D'^2-36} e^{2z} - \dfrac{1}{9D'^2-36} e^{0z}\right)$

$= \dfrac{1}{3}\left(z \cdot \dfrac{1}{18D' - \cancel{0}} e^{2z} - \dfrac{1}{9.0-36}\right)$

$= \dfrac{1}{3}\left(\dfrac{z e^{2z}}{18 \cdot 2} + \dfrac{1}{36}\right)$

$= \dfrac{1}{3}\left(\dfrac{z e^{2z}}{36} + \dfrac{1}{36}\right).$

$y = c_1 e^{2z} + c_2 e^{-2z} + \dfrac{z e^{2z}}{108} + \dfrac{1}{108}.$

$= c_1 (3x+2)^2 + c_2 \dfrac{1}{(3x+2)^2} + \dfrac{\log(3x+2)\,(3x+2)^2}{108} + \dfrac{1}{108}.$

Method of variation of Parameter.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + QY = X \longrightarrow (1).$$

Let, $y = c_1 u + c_2 V \longrightarrow (2)$ is the C.F

where, c_1 and c_2 are constants.

So, $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \longrightarrow (3)$ satisfies y.

i.e. $\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \longrightarrow (4)$

$$\frac{d^2V}{dx^2} + P\frac{dv}{dx} + QV = 0. \longrightarrow (5)$$

Let, $y = Au + BV \rightarrow (6)$ is the complete sol^n.

where, A and B are function.

$$\frac{dy}{dx} = AU_1 + A_1U + BV_1 + B_1V.$$

Let, A and B satisfy

$$A_1U + B_1V = 0 \longrightarrow (7)$$

than

$$\frac{dy}{dx} = AU_1 + BV_1$$

$$\frac{d^2y}{dx^2} = A_1U_1 + AU_2 + BV_2 + B_1V_1.$$

Put $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in given eqn.

$$A_1U_1 + AU_2 + BV_2 + B_1V_1 + P(AU_1 + BV_1) + Q(AU+ \\ = X$$

$\Rightarrow A(U_2 + PU_1 + QU) + B(V_2 + PV_1 + QV) + A_1U_1 + B_1V_1 =$

$\Rightarrow A_1U_1 + B_1V_1 = X \longrightarrow (8)$

Solving 7 and 8 we find A_1 and B_1.

$$A_1 = \frac{dA}{dx}, \quad B_1 = \frac{dB}{dx}.$$

Integrating A_1 and B_1 we find A and B.
Put A and B in ⑥ we get complete solⁿ.

12/08

(46) $\frac{d^2y}{dx^2} + n^2y = \sec nx$ by method of variation of Parameter.

Solⁿ :- Given eqn can be expressed -

$(D^2+n^2)y = \sec nx$.

A.E

$m^2+n^2 = 0$.

$\Rightarrow m = \pm ni$

$C.F = C_1 \cos nx + C_2 \sin nx$.

Let,

$y = A\cos nx + B\sin nx \rightarrow$ ① is the complete solⁿ

where A and B function of x.

$$\frac{dy}{dx} = A_1 \cos nx - nA\sin nx + B_1 \sin nx + nB\cos nx.$$

we choose A and B s.t

$A_1 \cos nx + B_1 \sin nx = 0 \longrightarrow$ ②.

then $\frac{dy}{dx} = -nA\sin nx + nB\cos nx$.

$$\frac{d^2y}{dx^2} = -n(A_1 \sin nx + An\cos nx)$$
$$+ n(B_1 \cos nx - nB\sin nx).$$

Put in given eqn.

$-n(A_1 \sin nx + An\cancel{\cos nx}) + n(B_1 \cos nx - nB\cancel{\sin nx})$

$+ n^2(A\cancel{\cos} nx + B\cancel{\sin} nx) = \sec nx$.

$\Rightarrow \cancel{A_1 n \sin nx + B}$

$\Rightarrow -A_1 n \sin nx + B_1 n \cos nx = \sec nx \longrightarrow (3)$

$A_1 \cos nx + B_1 \sin nx = 0 \longrightarrow (2)$

$(3) \times \cos nx + (2) \times n \sin nx$

$B_1 n \cos^2 nx + B_1 n \sin^2 nx = \sec nx \cos nx.$

$\Rightarrow B_1 n = 1$

$\Rightarrow B_1 = \frac{1}{n}.$

$\Rightarrow \frac{dB}{dx} = \frac{1}{n}.$

$\Rightarrow \int dB = \frac{1}{n} \int dx.$

$\Rightarrow B = \frac{1}{n} x + C_2$

Put B_1 in (2).

$A_1 \cos nx + \frac{1}{n} \sin nx = 0$

$\Rightarrow A_1 = -\frac{1}{n} \tan nx.$

$\Rightarrow \frac{dA}{dx} = -\frac{1}{n}. \tan nx.$

$\Rightarrow \int dA = -\frac{1}{n} \int \tan nx \, dx.$

$\Rightarrow A = -\frac{1}{n} \frac{\log|\sec nx|}{n} + C_1$

$= -\frac{\log|\sec x|}{n^2} + C_1.$

$\therefore y = A \cos nx + B \sin nx.$

$= \left(-\frac{\log|\sec x|}{n^2} + C_1\right) \cos nx + \left(\frac{1}{n} x + C_2\right) \sin nx$

(7) $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ by method of variation of Parameter.

$(D^2+4)y = 4\tan 2x.$

A. E.

$m^2+4=0$

$\Rightarrow m = \pm 2i$

$C.F = C_1 \cos 2x + C_2 \sin 2x$

Let, $y = A \cos 2x + B \sin 2x$. be the complete solⁿ.

where A and B are function of x.

$\frac{dy}{dx} = A_1 \cos 2x - 2A \sin 2x + B_1 \sin 2x + 2B \cos 2x.$

we choose A and B s.t

$A_1 \cos 2x + B_1 \sin 2x = 0.$

$\frac{dy}{dx} = -2A \sin 2x + 2B \cos 2x$

$\frac{d^2y}{dx^2} = -2(A_1 \sin 2x + 2A \cos 2x) + 2(B_1 \cos 2x - 2B \sin 2x)$

Put in given eqn.

$-2(A_1 \sin 2x + 2A\cancel{\cos 2x}) + 2(B_1 \cos 2x - 2B\cancel{\sin 2x}) + 4(A\cancel{\cos 2x} + B\cancel{\sin 2x})$

$= 4\tan 2x.$

$\Rightarrow -2A_1 \sin 2x + 2B_1 \cos 2x = 2\tan 2x \rightarrow (3)$

$A_1 \cos 2x + B_1 \sin 2x = 0 \rightarrow (2)$

$(3) \times \cos 2x + (2) \times \sin 2x$

$B_1 \cos^2 2x + B_1 \sin^2 2x = 2\tan 2x \cos 2x.$

$\Rightarrow B_1 = 2\sin 2x.$

$\Rightarrow \frac{dB}{dx} = 2\sin 2x.$

$\Rightarrow \int dB = 2\int \sin 2x \, dx$

$\Rightarrow B = -2 \frac{\cos 2x}{2} + C_2$

$= -\cos 2x + C_2.$

Put B_1 in ②.

$A_1 \cos 2x + 2\sin 2x \cdot \sin 2x = 0.$

$\Rightarrow A_1 = -\frac{2\sin^2 2x}{\cos 2x}.$

$\cancel{= 2\left(-\frac{2}{\cos 2x} - 2\cos x\right)} = -2\left(\frac{1-\cos^2 2x}{\cos 2x}\right)$

$= -2(\sec 2x - \cos 2x).$

$\frac{dA}{dx} = -2(\sec 2x - \cos 2x)$

$\int dA = -2\left[\int \sec 2x \, dx - \int \cos 2x \, dx\right]$

$\Rightarrow A = -2\left(\frac{\log \tan \frac{2x}{2}}{2} - \frac{\sin 2x}{2}\right) + C_1$

$= -\log \tan x + \sin 2x. + C_1$

$y = (-\log \tan x + \sin 2x + C_1)\cos 2x + (-\cos 2x + C_2)\sin$

(48) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$. by method of variation of Parameter.

$(x^2D^2 + xD - 1)y = x^2e^x \longrightarrow (1)$

Put $x = e^z$, $\log x = z$.

$xD = D'$ $\qquad D' = \frac{d}{dz}$.

$x^2D^2 = D'(D'-1)$

Put in (1)

$(D'(D'-1) + D' - 1)y = e^{2z}.e^{e^z}$.

$(D'^2 - 1)y = e^{2z}.e^{e^z}$.

A.E

$m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$\therefore C.F = C_1e^z + C_2e^{-z}$.

$= C_1x + C_2\frac{1}{x}$

Let,

~~$y = Ae^x +$~~

$y = Ax + B\frac{1}{x}$ be the complete soln.

$\frac{dy}{dx} = A_1x + A + B_1\frac{1}{x} - \frac{B}{x^2}$ where A and B are fn of x.

we choose A and B s.t

$A_1x + B_1\frac{1}{x} = 0$.

$\therefore \frac{dy}{dx} = A - \frac{B}{x^2} = 0$

$\frac{d^2y}{dx^2} = A_1 - \frac{B_1}{x^2} + \frac{2B}{x^3}$

Put in given eqn.

$$x^2\left(A_1 - B_1\frac{1}{x^2} + B\cdot\frac{2}{x^3}\right) + x\left(\cancel{A} - B\cancel{\frac{1}{x^2}}\right)$$
$$-\left(A\cancel{x} + \cancel{\frac{B}{x}}\right) = x^2e^x.$$

$$\Rightarrow A_1x^2 - B_1 = x^2e^x. \quad \text{---} (1)$$
$$A_1x + \frac{B_1}{x} = 0.$$
$$\Rightarrow A_1x^2 + B_1 = 0. \quad \text{---} (2)$$

(1) + (2).

$$2A_1x^2 = x^2e^x.$$
$$\Rightarrow 2A_1 = e^x.$$
$$\Rightarrow \frac{dA}{dx} = \frac{e^x}{2}$$
$$\Rightarrow A = \frac{e^x}{2} + C_1$$

Put A_1 in (2).

$$\frac{e^x}{2}\cdot x^2 + B_1 = 0$$
$$\Rightarrow B_1 = -\frac{1}{2}e^x\cdot x^2$$
$$\Rightarrow \int dB = -\frac{1}{2}\int e^x\cdot x^2\,dx.$$
$$\Rightarrow B = -\frac{1}{2}\left[x^2\cdot e^x - \int 2x\,e^x\,dx\right]$$
$$= -\frac{1}{2}\left[x^2\cdot e^x - \{2x\cdot e^x - 2e^x\}\right] + C_2$$
$$= -\frac{1}{2}x^2e^x + xe^x - e^x + C_2.$$

$$y = Ax + \frac{1}{x}B$$
$$= \left(\frac{e^x}{2} + C_1\right)x + \left(-\frac{1}{2}x^2e^x + xe^x - e^x + C_2\right)\frac{1}{x}$$

Principal of Superposition for Homogeneous Equation

6.1 Defination: Principal of Superposition for Homogeneous Equation is defined a the linear combination of any solution of a homogeneous linear differential equation of order two is also a solution of the givin differential equation.

Note: This principle is applicable only for homogeneous linear ODE.

Theorem 1. If y_1 and y_2 are two solution of homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \text{ on interval I}$$

Then $y = c_1y_1 + c_2y_2$ is also solution of $y'' + p(x)y' + q(x)y = 0$ on interval I

Proof: Let y_1 and y_2 are two solution of homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad \text{.......(1)}$$

on interval I then they must satisfy the equation (1)

Then $y_1'' + p(x)y_1' + q(x)y_1 = 0$ ....(2)

and $y_2'' + p(x)y_2' + q(x)y_2 = 0$ ....(3)

if c_1 and c_2 are constant let

$$y = c_1y_1 + c_2y_2 \quad \text{.......(4)}$$

on differentiating

$$y' = c_1y_1' + c_2y_2' \text{ and } y'' = c_1y_1'' + c_2y_2'' \quad \text{.......(5)}$$

with the help of equation (4) and (5) equation (1) becomes

$$y'' + p(x)y' + q(x)y = \{c_1y_1'' + c_2y_2''\} + \{p(x)(c_1y_1' + c_2y_2')\} + \{q(x)(c_1y_1 + c_2y_2)\}$$

$$= c_1\{y_1'' + p(x)y_1' + q(x)y_1\} + c_2\{y_2'' + p(x)y_2' + q(x)y_2\}$$

$\Rightarrow \quad y'' + p(x)y' + q(x)y = c_1 0 + c_2 0$ using equation (2) and (3)

$\Rightarrow \quad y'' + p(x)y' + q(x)y = 0$

Thus $y = c_1y_1 + c_2y_2$ also satisfy (1)

Hence $y = c_1y_1 + c_2y_2$ is the solution of

$y'' + p(x)y' + q(x)y = 0$ on interval I.

Example 1. Show that $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are two solution of the equation $y'' - 9y = 0$ also verify the principal of super position.

Solution: Givin equation is

$$y'' - 9y = 0 \quad \text{.........}(1)$$

Now let $y(x) = e^{3x}$

$\therefore \quad y'(x) = 3e^{3x}$

$y''(x) = 9e^{3x}$

Now putting thease values in (1)

$$y'' - 9y = 9e^{3x} - 9e^{3x} = 0$$

Hence e^{3x} is the solution of (1)

Now again let $y = e^{-3x}$

$\therefore \quad y'(x) = -3e^{-3x}$

$$y''(x) = 9e^{-3x}$$

Now putting thease value in (1)

$$y'' - 9y = 9e^{-3x} - 9e^{-3x} = 0$$

Hence e^{-3x} is the solution of (1)

Thus $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are two solution of the differential equation (1)

Verify

$y = c_1e^{3x} + c_2e^{-3x}$ is also the solution of (1)

Thus

$$y'' - 9y = \left(c_1e^{3x} + c_2e^{-3x}\right)'' - 9\left(c_1e^{3x} + c_2e^{-3x}\right)$$

$$= c_1 9e^{3x} + c_2 9e^{-3x} - 9c_1e^{3x} + 9c_2e^{-3x} = 0$$

Thus

$y = c_1e^{3x} + c_2e^{-3x}$ is general solution of $y^{II} - 9y = 0$. Hence principle of superposition is verify.

Example 2. Show that y_1 = cosx and y_2 = sinx are two solution of the equation $y'' + y = 0$ also verify the principal of super position.

Solution: Givin equation is

$$y'' + y = 0 \quad \ldots\ldots (1)$$

Now let y = cosx

$\therefore \quad y' = -\sin x$

and $\quad y'' = -\cos x$

putting thease values in (1)

$\therefore \quad y'' + y = -\cos x + \cos x = 0$

Hence y = cosx is the solution of (1)

Now again let $y = \sin x$

$$\therefore \quad y' = \cos x$$

and $\quad y'' = -\sin x$

Hence y = sinx is the solution of (1)

Verify

$y = c_1 \cos x + c_2 \sin x$ is also the solution of (1)

Thus

$$y'' + y = (c_1 \cos x + c_2 \sin x)'' - 9(c_1 \cos x + c_2 \sin x)$$

$$= -c_1 \cos x - c_2 \sin x + c_2 \cos x + c_2 \sin x = 0$$

Thus

$y = c_1 \sin x + c_2 \sin x$ is general solution of (1).

Hence principle of superposition is verify.

Example 3. Show that $y_1 = x^2$ and $y_2 = 1$ are two solution of the homogeneous non linear ordinary differential equation $y''y - xy' = 0$ also verify the principal of super position.