Infinite Series

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Definition Given a sequence (a_n) of real numbers, a formal sum of the form $\sum_{n=1}^{\infty} a_n$ (or $\sum a_n$, for short) is called an *infinite series.*

For any $n \in \mathbb{N}$, the finite sum $s_n := a_1 + \cdots + a_n$ is called the (n-th) *partial sum* of the series $\sum a_n$.

A more formal definition of an infinite series is as follows. By the symbol $\sum_n a_n$ we mean the sequence (s_n) where $s_n := a_1 + \cdots + a_n$.

Convergent Series :

A series $\Sigma\, u_n$ is said to be convergent if S_n, the sum of its first n terms, tends to a definite finite limit S as n tends to infinity.

We write $S = \lim_{n \to \infty} S_n$.

The finite limit S to which S_n tends is called the sum of the series.

Divergent Series: *A series $\Sigma\, u_n$ is said to be divergent if S_n, the sum of its first n terms, tends to either $+\infty$ or $-\infty$ as n tends to infinity,*

i.e., if $\lim_{n \to \infty} S_n = \infty$ *or* $-\infty$.

Oscillatory Series: *A series $\Sigma\, u_n$ is said to be an oscillatory series if S_n, the sum of its first n terms, neither tends to a definite finite limit nor to $+\infty$ or $-\infty$ as n tends to ∞.*

The series is said to *oscillate finitely,* if the value of S_n as $n \to \infty$ fluctuates within a finite range. It is said to *oscillate infinitely,* if S_n tends to infinity and its sign is alternately positive and negative.

Sequence of Partial Sums of a Series :

If S_n denotes the sum of the first n terms of the series $\Sigma\, u_n$, so that

$$S_n = u_1 + u_2 + \ldots + u_n,$$

then S_n is called the partial sum of the first n terms of the series and the sequence $< S_n > = < S_1, S_2, \ldots, S_n, \ldots >$ is called the **sequence of partial sums** of the given series. We can define the convergent, divergent and oscillatory series in terms of the sequence of partial sums.

Definition: *A series $\Sigma\, u_n$ is said to be convergent, divergent or oscillatory according as the sequence $< S_n >$ of its partial sums is convergent, divergent or oscillatory.*

If the sequence $< S_n >$ of partial sums of a series $\Sigma\, u_n$ converges to S then S is said to be the sum of the series $\Sigma\, u_n$.

Series

$U_1 + U_2 + U_3 + \ldots \infty = \sum_{n=1}^{\infty} U_n$ is called infinite series

$S_n = U_1 + U_2 + \ldots + U_n$ is n^{th} Partial sum.

$\lim_{n \to \infty} S_n = S$ (a finite no), then the given series is called convergent series.

$\lim_{n \to \infty} S_n = \pm\infty$, then the given series is called divergent series.

Ex-7 Test the convergency of the following series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots \infty.$$

Soln:-

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{n\left(1 + \frac{1}{n}\right)}$$

$$= \frac{1}{1+0}$$

$= 1$, a finite no.

Hence, given series is convergent.

Ex-8 Test the Convergency -

$$1^2 + 2^2 + 3^2 + \dots + \infty.$$

Soln:- $S_n = 1^2 + 2^2 + \dots + n^2.$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6}$$

$$= \infty.$$

Hence, the given series is divergent.

Ex-9 Test the Convergency.

$$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \dots \infty.$$

Soln:-

$$S_n = \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \dots + \frac{1}{2n \cdot (2n+2)}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots + \left(\frac{1}{2n} - \frac{1}{2n+2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2n+2}\right]$$

$$= \frac{n}{2(2n+2)}$$

$$\therefore \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{2(2n+2)}$$

$$= \lim_{n \to \infty} \frac{n}{4n\left(1 + \frac{1}{n}\right)}$$

$$= \frac{1}{4(1+0)} = \frac{1}{4},$$ a finite no.

Hence, given series is convergent.

G.P series.

$a + ar + ar^2 + ar^3 + \dots \infty$ is a infinite G.P series.

$S_n = a + ar + ar^2 + \dots + ar^{n-1}$

$= \frac{a(1-r^n)}{1-r}$

Case I

$0 < r < 1$.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

$= \frac{a(1 - \lim_{n\to\infty} r^n)}{1-r}$

$= \frac{a(1-0)}{1-r}$

$= \frac{a}{1-r}$, a finite no.

Hence, the given series is convergent for $0<r<1$

Case II

$r > 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

$= \frac{a(1 - \lim_{n\to\infty} r^n)}{1-r}$

$= \infty$..

So, the given series is divergent for $r > 1$

Case III

$r \leq 0$

Then, the series is oscillatory.

Ex-10 Test the Convergency.

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\infty.$$

Soln:- This is infinite GP series. with common ratio

$$r=\frac{1}{3}<1.$$

So, Given series is Convergent.

Comparison Test

Σu_n is a positive series and Σv_n is another Positive series.

If $u_n \leq v_n k$ or $\lim\limits_{n\to\infty}\frac{u_n}{v_n}$ = finite no, ~~then~~

if Σv_n is convergent, than Σu_n is also convergent.

Σv_n is divergent, than Σu_n is also divergent.

P- series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots\infty$$ is called P-series.

Prove that P series is convergent. if $P>1$ and divergent if $P\leq 1$.

Proof:- $$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots\infty.$$

Case I

$P>1$

$$2^p<3^p \Rightarrow \frac{1}{2^p}>\frac{1}{3^p}$$

$$\Rightarrow \frac{1}{2^p}+\frac{1}{2^p}>\frac{1}{2^p}+\frac{1}{3^p}$$

$$\Rightarrow \frac{2}{2^p}>\frac{1}{2^p}+\frac{1}{3^p}.$$

Again,

$$4^p < 5^p \Rightarrow \frac{1}{4^p} > \frac{1}{5^p}$$

$$4^p < 6^p \Rightarrow \frac{1}{4^p} > \frac{1}{6^p}$$

$$4^p < 7^p \Rightarrow \frac{1}{4^p} > \frac{1}{7^p}$$

Adding

$$\frac{3}{4^p} > \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}$$

$$\frac{4}{4^p} > \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}$$

Ily

$$\frac{8}{8^p} > \frac{1}{8^p} + \frac{1}{9^p} + \dots + \frac{1}{15^p}$$

- - - -
- . .

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots \infty$$

$$\leq 1 + \frac{2}{2^p} + \frac{4}{4^p} + \dots \infty \quad — (1)$$

$1 + \frac{2}{2^p} + \frac{4}{4^p} + \dots \infty$ is infinite G.P series.

with common ratio

$$r = \frac{2}{2^p}$$

$$= 2^{1-p} < 1 \quad \left[\because p > 1 \Rightarrow 0 > 1-p \right]$$

So, given p- series is convergent.

Case II

$p = 1.$

$$\sum \frac{1}{n^p} = \sum \frac{1}{n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots \infty.$$

which is obviously divergent.

$U_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

$U_{n+p} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1} + \dots +$

$U_{n+p} - U_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p}$

$U_{n+p} - U_n > \frac{1}{2n} + \frac{1}{2n} \dots + \frac{1}{2n}$

$U_{n+p} - U_n > \frac{n}{2n} = \frac{1}{2}$

$|U_{n+p} - U_n| > \frac{1}{n}.$

$\therefore U_n$ is divergent.

Case III

$p < 1$

$n^p < n$.

$\Rightarrow \frac{1}{n^p} > \frac{1}{n}$.

$\Rightarrow \sum \frac{1}{n^p} > \sum \frac{1}{n}$.

$\Rightarrow \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \infty$

which is divergent.

So, $\sum \frac{1}{n^p}$ is also divergent.

So, Given series is convergent for $p > 1$ and divergent for $p \leq 1$.

Ex-11 Test the convergency of the series —

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \dots \frac{2n-1}{n(n+1)(n+2)} + \dots \infty.$$

Soln:- $U_n = \frac{2n-1}{n(n+1)(n+2)}$

Consider a series

$\sum v_n$, $v_n = \frac{n}{n^3} = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{v_n} = \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} \cdot n^2.$$

$$= \lim_{n\to\infty} \frac{n(2-1/n)}{n\cdot n(1+\frac{1}{n})\cdot n(1+2/n)} \cdot n^2$$

$$= \frac{(2-0)}{(1+0)(1+0)} = 2, \text{ a finite no.}$$

$\sum v_n = \sum \frac{1}{n^2}$ is a convergent series because it is p-series where $p = 2 > 1$

So, by Comparison test.
Given series is convergent.

Ex-12 Test the convergency of the series.

$$\frac{6}{1\cdot3\cdot5} + \frac{8}{3\cdot5\cdot7} + \frac{10}{5\cdot7\cdot9} + \cdots \infty.$$

Solⁿ:- $U_n = \dfrac{(2n+4)}{(2n-1)(2n+1)(2n+3)}$

consider a series ΣV_n
where, $V_n = \dfrac{n}{n^3} = \dfrac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{2n+4}{(2n-1)(2n+1)(2n+3)} \cdot n^2.$$

$$= \lim_{n\to\infty} \frac{2n(1+2/n)}{n(2-\frac{1}{n})n(2+\frac{1}{n})n(2+3/n)} \cdot n^2$$

$$= \frac{2\cdot(1+0)}{(2-0)(2+0)(2+0)}$$

$$= \frac{2}{8}$$

$= \frac{1}{4}$, a finite no.

$\Sigma V_n = \Sigma \frac{1}{n^2}$ is convergent series because it is a P-series with $P = 2 > 1$.

So, by comparison test ΣU_n is convergent series.

Ex-13 Test the convergency of -

S-15 $\frac{1}{1\cdot 3}+\frac{2}{3\cdot 5}+\frac{3}{5\cdot 7}+\cdots$

Soln:- $U_n = \frac{n}{(2n-1)(2n+1)}$

Consider a series Σv_n.

$$v_n = \frac{n}{n^2} = \frac{1}{n}.$$

$$\lim_{n\to\infty}\frac{U_n}{v_n} = \lim_{n\to\infty}\frac{n}{(2n-1)(2n+1)}\cdot n$$

$$= \lim_{n\to\infty}\frac{n\cdot n}{n(2-1/n)\,n(2+\frac{1}{n})}$$

$$= \frac{1}{(2-0)(2+0)} = \frac{1}{4},$$ a finite no.

$\Sigma v_n = \Sigma\frac{1}{n}$ is divergent because it is a P-series

~~with~~ and $P=1$

So, by comparison test ΣU_n is divergent series.

Ex-14 Test the convergency of the series -

$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots\ \infty.$

Soln:- $U_n = \frac{1}{n\cdot(n+1)(n+2)}$

Consider a series Σv_n.

$$v_n = \frac{1}{n^3}.$$

$$\lim_{n\to\infty}\frac{U_n}{v_n} = \lim_{n\to\infty}\frac{1}{n(n+1)(n+2)}\cdot n^3.$$

$$= \lim_{n\to\infty}\frac{n^3}{n\cdot n(1+1/n)\cdot n(1+2/n)}$$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(1+0)(1+0)}$

$= 1$, a finite no.

$\therefore \sum v_n = \sum \frac{1}{n^3}$ is convergent because it is a p-series and $p = 3 > 1$.

So, by comparison test $\sum u_n$ is convergent series

Ex-15 Test the convergency of the series—

$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$

Soln:-

$u_n = \frac{n^n}{(n+1)^{n+1}}$ (Neglecting 1st term).

Consider a series, $\sum v_n$

where, $v_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \times n.$

$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \cdot \frac{n}{(n+1)}$

$= \lim_{n\to\infty} \frac{1}{\frac{(n+1)^n}{n^n}} \cdot \lim_{n\to\infty} \frac{n}{n+1}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} \lim_{n\to\infty} \frac{n}{n(1+\frac{1}{n})}$

$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} \cdot \frac{1}{(1+0)}$

$= \frac{1}{e} \cdot 1 = \frac{1}{e}$

a finite no.

$\because \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$ or $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

$\sum v_n = \sum \frac{1}{n}$ is divergent because it is a P-series and $P = 1$.

So, by comparison test $\sum U_n$ is divergent series.

D'Alembert's Ratio test

$\sum U_n$ be a series of Positive term and $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$

Given series is convergent when $l < 1$

" " " divergent " $l > 1$

when $l = 1$, test fails.

Ex-16. Test the convergency of the series-

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots \infty.$$

Soln:- $U_n = \frac{x^n}{n^2+1}$ (neglecting 1st term).

$$U_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{x^n}$$

$$= \lim_{n\to\infty} \frac{x^{n+1}}{(n^2+2n+1)+1} \cdot \frac{n^2\left(1+\frac{1}{n^2}\right)}{x^n}$$

$$= \lim_{n\to\infty} \frac{x \cdot n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}$$

$$= \frac{x \cdot (1+0)}{(1+0+0)}$$

$$= x.$$

By D'Alembert's ratio test, given series is convergent when $x < 1$ and divergent when $x > 1$ and when $x = 1$, test fails.

Now,

$x = 1$

series is

$$1 + \frac{1}{2} + \frac{1}{5} + \cdots$$

$$U_n = \frac{1}{n^2+1}$$

consider a series $\sum V_n$.

$$V_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{n^2+1} \cdot n^2.$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \frac{1}{1+0} = 1, \text{ a finite no.}$$

~~So, $\sum U_n$~~ So, $\sum V_n = \frac{1}{n^2}$ is convergent because it is a P- series and $P = 2 > 1$.

So, by comparison test $\sum U_n$ is convergent.

Hence,

$\sum U_n$ convergent $x \leq 1$

$\sum U_n$ divergent $x > 1$.

Ex-17 Test the convergency of the series. -

$$2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{n+1}{n^3} x^n + \cdots \infty$$

Soln:- $U_n = \frac{n+1}{n^3} \cdot x^n$

$$U_{n+1} = \frac{n+1+1}{(n+1)^3} \cdot x^{n+1}$$

$$\lim_{n\to\alpha} \frac{U_{n+1}}{U_n} = \lim_{n\to\alpha} \frac{n+2}{(n+1)^3} \cdot x^{n+1} \cdot \frac{n^3}{(n+1)\cdot x^n}.$$

$$= \lim_{n\to\alpha} \frac{n(1+2/n)\cdot x \cdot n^3}{n^3(1+\frac{1}{n^3})^3 \cdot n(1+\frac{1}{n})}$$

$$= \frac{(1+0)\cdot x \cdot}{(1+0)(1+0)}$$

$$= x.$$

Given, series is convergent if $x < 1$
divergent if $x > 1$

and when $x=1$, test fails

for, $x=1$, the series is -

$$2 + \frac{3}{8} + \frac{4}{27} + \cdots$$

$$U_n = \frac{n+1}{n^3}.$$

Consider a series ΣV_n,

where, $V_n = \frac{n}{n^3} = \frac{1}{n^2}$

$$\lim_{n\to\alpha} \frac{U_n}{V_n} = \lim_{n\to\alpha} \frac{n+1}{n^3} \cdot n^2$$

$$= \lim_{n\to\alpha} \frac{n(1+1/n)\cdot n^2}{n^3}$$

$$= (1+0) = 1,$$ a finite no.

$\therefore$ $\Sigma V_n = \Sigma \frac{1}{n^2}$ is convergent because it is a p-series

and $p = 2 > 1$.

So, by comparison test ΣU_n is convergent.

So, Σ ~~U_n convergent for $n \leq 1$~~

$\therefore$ ΣU_n is convergent for $x \leq 1$
" " divergent " $x > 1$

Ex-18 Test the convergency of the series-

$$\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\cdots \infty$$

Soln:- $U_n = \frac{x^n}{n}$

$$U_{n+1} = \frac{x^{n+1}}{n+1}$$

Now,

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^{n+1}}{n+1} \times \frac{n}{x^n}.$$

$$= \lim_{n\to\infty} \frac{x}{n(1+\frac{1}{n})} \times n.$$

$$= \lim_{n\to\infty} \frac{x}{1+\frac{1}{n}}$$

$$= \frac{x}{1+0} = x.$$

So, by D'E Alemberts ratio test, given series is

convergent if $x<1$.

divergent if $x>1$.

and the test fails $x=1$.

If $x=1$., the series is-

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \infty$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ is a P- series and $P=1$

So, it is divergent.

So, the given series is convergent for $x<1$

divergent for $x \geq 1$.

Ex. 19 Test the convergency of the series.

$$\frac{2}{1^2}x + \frac{3}{2^2}x^2 + \frac{4}{3^2}x^3 + \dots \infty.$$

Solⁿ :- $\quad U_n = \dfrac{(n+1)}{n^2} x^n$

$$U_{n+1} = \frac{(n+2)}{(n+1)^2} \cdot x^{n+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)^2} \cdot x^{n+1} \cdot \frac{n^2}{(n+1)\cdot x^n}$$

$$= \lim_{n\to\infty} \frac{n(1+2/n) \cdot x \cdot n^2}{n^2(1+\frac{1}{n^2})^2 \cdot n(1+\frac{1}{n})}$$

$$= \frac{x(1+0)}{(1+0)^2 \cdot (1+0)}$$

$$= x.$$

So, by D'Alembert's test

given series is convergent if $x<1$

divergent if $x>1$

test fails if $x=1$

If $x=1$, the series becomes.

$$\frac{2}{1^2} + \frac{3}{2^2} + \frac{4}{3^2} + \dots$$

$$U_n = \frac{n+1}{n^2}$$

consider a series ΣV_n, $V_n = \dfrac{n}{n^2} = \dfrac{1}{n}$

Now, $\lim_{n\to\infty} \dfrac{U_n}{V_n} = \lim_{n\to\infty} \dfrac{n+1}{n^2} \cdot n$.

$$= \lim \frac{n(1+1/n)\cdot n}{n^2}$$

$= (1+0) = 1$, a finite no.

$\Sigma v_n = \Sigma \frac{1}{n}$ is divergent because it is a P-series with $P = 1$

So, by comparison test Σu_n is also divergent

Hence, the given series is convergent if $x < 1$
divergent if $x \geq 1$.

Raabe's Test

$\sum U_n$ be a series of Positive term.

$$\lim_{n \to \infty} \left[n \left(\frac{U_n}{U_{n+1}} - 1 \right) \right] = l .$$

If $l > 1$, series is convergent.

$l < 1$, " " divergent.

$l = 1$, test fails.

Ex-21 Test the convergency of the series -

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^2}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^3}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^4}{7} + \dots \infty.$$

Solⁿ. -

$$U_n = \frac{1\cdot 3\cdot 5 \dots (2n-1)}{2\cdot 4\cdot 6 \dots 2n}\cdot\frac{x^{n+1}}{2n+1}$$ (Neglecting 1st term)

$$U_{n+1} = \frac{1\cdot 3\cdot 5 \dots (2n-1)\cdot(2n+1)}{2\cdot 4\cdot 6 \dots 2n \cdot (2n+2)}\cdot\frac{x^{n+2}}{2n+3}.$$

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\frac{1\cdot 3\cdot 5 \dots (2n-1)(2n+1)\cdot x^{n+2}}{2\cdot 4\cdot 6 \dots 2n\cdot(2n+2)\cdot(2n+3)}\cdot\frac{2\cdot 4\cdot 6 \dots 2n\,(2n+1)}{1\cdot 3\cdot 5 \dots (2n-1)\,x^{n+1}}$$

$$= \lim_{n\to\infty}\frac{\cancel{x^{n+2}}\,(2n+1)(2n+1)}{(2n+2)(2n+3)}\cdot x.$$

$$= \lim_{n\to\infty}\frac{n(2+1/n)\,n(2+1/n)}{n(2+2/n)\,n(2+3/n)}\cdot x.$$

$$= \frac{(2+0)(2+0)}{(2+0)(2+0)}\cdot x.$$

$$= x.$$

So, by D' Alembert's ratio test, the given series is convergent when $x < 1$
divergent when $x > 1$
and when $x = 1$ test fails.

If $x = 1$, then the series is

$$1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{7} + \dots \infty.$$

$$U_n = \frac{1\cdot 3\cdot 5 \dots (2n-1)}{2\cdot 4\cdot 6 \dots 2n}\cdot\frac{1}{2n+1}$$

Here Comparison test fails.

Now,

$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}$$

$$\lim_{n\to\infty}\left[n\left(\frac{U_n}{U_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left\{\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}-1\right\}\right]$$

$$= \lim_{n\to\infty}\left[n\ \frac{4n^2+6n+4n+6}{4n^2+4n+1}-1\right]$$

$$\cancel{= \lim_{n\to\infty}\left[n \cdot \frac{n^2(4+10/n+6/n^2)}{n^2}\right.}$$

$$= \lim_{n\to\infty}\left[n \cdot \frac{4n^2+10n+6-4n^2-4n-1}{4n^2+4n+1}\right]$$

$$= \lim_{n\to\infty}\left[n \cdot \frac{6n+5}{4n^2+4n+1}\right]$$

$$= \lim_{n\to\infty}\left[n \cdot \frac{n(6+5/n)}{n^2(4+4/n+\frac{1}{n^2})}\right]$$

$$= \frac{(6+0)}{(4+0+0)}$$

$$= 3/2 > 1.$$

So, by Raabe's Test, given series is Convergent when $x=1$

Hence, given series is Convergent when $x \leq 1$.

divergent when $1 < x$.

Ex-22 Test the convergency of the series —

S-15

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^7}{7} + \cdots \infty.$$

Soln:-

$$u_n = \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots 2n}\cdot\frac{x^{2n+1}}{2n+1}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots 2n\,(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}.$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)\cdot x^{2n+3}}{2\cdot 4\cdot 6\cdots 2n\,(2n+2)\cdot(2n+3)}\cdot\frac{2\cdot 4\cdot 6\cdots 2n\,(2n+1)}{1\cdot 3\cdot 5\cdots(2n-1)\,x^{2n+1}}$$

$$= \lim_{n\to\infty}\frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}\cdot x^2.$$

$$= \lim_{n\to\infty}\frac{n(2+1/n)\,n(2+1/n)}{n(2+2/n)\,n(2+3/n)}\cdot x^2.$$

$$= \frac{(2+0)(2+0)}{(2+0)(2+0)}\cdot x^2$$

$$= x^2.$$

$\therefore$ By D'Alembert's ratio test given series is

convergent when $x^2 < 1$.

$\Rightarrow -1 < x < 1$.

divergent when $1 < x^2$

$\Rightarrow x < -1$ or $1 < x$.

when $x^2 = 1$

$\Rightarrow x = \pm 1$ test fails.

when, $x = \pm 1$, the series becomes.

$$1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \cdots$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

$$\lim_{n\to\infty}\left[n\left(\frac{U_n}{U_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n \cdot \frac{4n^2+10n+6}{4n^2+4n+1}-1\right]$$

$$= \lim_{n\to\infty}\left[n \cdot \frac{6n+5}{n^2(4+4/n+\frac{1}{n^2})}\right]$$

$$= \lim_{n\to\infty}\left[n \cdot \frac{n(6+5/n)}{n^2(4+4/n+\frac{1}{n^2})}\right]$$

$$= \frac{6}{4}$$

$$= \frac{3}{2} > 1.$$

So, by Raabe's test, the given series is ~~conv~~ convergent when $x^2=1 \Rightarrow x = \pm 1$.

So, the given series is convergent for $x^2 \leq 1$
divergent for $x^2 > 1$.

~~Cauchy~~ Cauchy's Root test -

$\sum U_n$ be a series of Positive term, if $\lim_{n\to\infty}(U_n)^{\frac{1}{n}} < 1$, given series is convergent and $\lim_{n\to\infty}(U_n)^{\frac{1}{n}} > 1$, given series is divergent and when equal to 1, test fails.

(Cauchy root test is applicable for Power series

Ex-23 Test the Convergency of the series ΣU_n.

S-15 Where $U_n = \dfrac{n^{n^2}}{(1+n)^{n^2}}$

Soln.- $\lim_{n\to\infty} (U_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{ \dfrac{n^{n^2}}{(1+n)^{n^2}} \right\}^{\frac{1}{n}}$.

$= \lim_{n\to\infty} \dfrac{n^n}{(1+n)^n}$

$= \lim_{n\to\infty} \dfrac{1}{\left(\frac{1+n}{n}\right)^n}$

$= \lim_{n\to\infty} \dfrac{1}{\left(1+\frac{1}{n}\right)^n}$.

$= \dfrac{1}{e} < 1$.

So, by Cauchy root test, given series is convergent.

Ex-24 Test the Convergency of the series -

$\left(\dfrac{2^2}{1^2} - \dfrac{2}{1}\right)^{-1} + \left(\dfrac{3^3}{2^3} - \dfrac{3}{2}\right)^{-2} + \left(\dfrac{4^4}{3^4} - \dfrac{4}{3}\right)^{-3} + \dots \infty$

Soln:- $\cancel{U_n = \dfrac{(n+1)^{n+1}}{(n)^{n+1}}}$ $U_n = \left\{ \left(\dfrac{n+1}{n}\right)^{n+1} - \dfrac{n+1}{n} \right\}^{-n}$

$\lim_{n\to\infty} (U_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{ \left(\dfrac{n+1}{n}\right)^{n+1} - \dfrac{n+1}{n} \right\}^{-n \cdot \frac{1}{n}}$

$= \lim_{n\to\infty} \left\{ \left(\dfrac{n+1}{n}\right)^{n+1} - \dfrac{n+1}{n} \right\}^{-1}$

$= \lim_{n\to\infty} \left\{ \left(1+\dfrac{1}{n}\right)^n \left(1+\dfrac{1}{n}\right) - \left(1+\dfrac{1}{n}\right) \right\}^{-1}$

$= (e \cdot 1 - 1)^{-1}$.

$= \dfrac{1}{e-1} < 1$.

So, by Cauchy's root test given series is Convergent.

Ex-26 Test the Convergency

$$1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots \infty$$

Soln:-

$$U_n = \frac{1}{\underline{|n}}$$

$$U_{n+1} = \frac{1}{\underline{|n+1}}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\underline{|n}}{\underline{|n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1$$

So, by D'Alembert's test given series is convergent

Ex-26 Test the Convergency

$$1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots \infty$$

Soln:- $U_n = \frac{1}{\lfloor n}$

$$U_{n+1} = \frac{1}{\lfloor n+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\lfloor n}{\lfloor n+1}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1$$

So, by D'Alembert's test given series is convergent

Cauchy's general Principle of Convergency. (Proof on last)

Statement :- A necessary and sufficient condition for the Convergency of the sequence $\langle U_n \rangle$ is that for $\epsilon > 0$ there exist $m \geq 0$. such that $|U_{n+p} - U_m| < \epsilon$ for $n \geq m$, $p > 0$.

Some important result on limit :-

① If $\langle U_n \rangle$ be a sequence such that.

$$\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = l, \quad \text{where } 0 \leq l < 1$$

then $\lim_{n\to\infty} U_n = 0$.

Ex-27 Prove that $\lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0$.

Solⁿ:-

$$U_n = \frac{x^n}{\lfloor n}$$

$$U_{n+1} = \frac{x^{n+1}}{\lfloor n+1}$$

$$\cancel{\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^n}{\lfloor n} \cdot \frac{\lfloor n+1}{x^{n+1}} = n+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^{n+1}}{\lfloor n+1} \cdot \frac{\lfloor n}{x^n}$$

$$= \lim_{n\to\infty} \frac{x}{n+1}$$

$$= \frac{1}{\infty} = 0$$

$$\therefore \lim_{n\to\infty} U_n = 0.$$

$$\therefore \lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0.$$

Ex-28 If $U_n = \frac{3\cdot5\cdot7\cdots(2n+1)}{2\cdot5\cdot8\cdots(3n-1)}$. Prove that $\lim_{n\to\infty} U_n = 0$.

Solⁿ:- $U_{n+1} = \frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)} \cdot \frac{2\cdot5\cdot8\cdots(3n-1)}{3\cdot5\cdot7\cdots(2n+1)}$$

$$= \lim_{n\to\infty} \frac{2n+3}{3n+2}$$

$$= \lim_{n\to\infty} \frac{n(2+3/n)}{n(3+2/n)}$$

$$= 2/3 < 1$$

$$\therefore \lim_{n\to\infty} U_n = 0 \Rightarrow \cancel{\lim_{n\to\infty}}$$

② If $\langle U \rangle$ be a sequence of positive term

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l, \text{ then } \lim_{n\to\infty} U_n^{\frac{1}{n}} = l.$$

Ex-29 Find the $\lim_{n\to\infty} \left(\frac{\lfloor 2n}{\lfloor n \lfloor n}\right)^{\frac{1}{n}}$.

Soln:- Let,

$$U_n = \frac{\lfloor 2n}{\lfloor n \lfloor n}$$

$$U_{n+1} = \frac{\lfloor 2n+2}{\lfloor n+1 \lfloor n+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\lfloor 2n+2}{\lfloor n+1 \lfloor n+1} \cdot \frac{\lfloor n \lfloor n}{\lfloor 2n}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{n(2+2/n)\,n(2+1/n)}{n(1+1/n)\,n(1+1/n)}$$

$$= \frac{(2+0)(2+0)}{(1+0)(1+0)}$$

$$= 4$$

$$\therefore \lim_{n\to\infty} (U_n)^{\frac{1}{n}} = 4$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{\lfloor 2n}{\lfloor n \lfloor n}\right)^{\frac{1}{n}} = 4.$$

Ex-30 Prove that $\lim_{n\to\infty} \left(\frac{\lfloor 3n}{\lfloor n \lfloor n \lfloor n}\right)^{\frac{1}{n}} = 27$.

Soln:- Let,

$$U_n = \frac{\lfloor 3n}{\lfloor n \lfloor n \lfloor n}$$

$$U_{n+1} = \frac{\lfloor 3n+3}{\lfloor n+1 \lfloor n+1 \lfloor n+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{\lfloor 3n+3}{\lfloor n+1 \lfloor n+1 \lfloor n+1} \cdot \frac{\lfloor n \lfloor n \lfloor n}{\lfloor 3n}$$

$$\frac{(3n+3)(3n+2)(3n+1)}{(n+1)(n+1)(n+1)}$$

$$\lim_{n\to\alpha} \frac{U_{n+1}}{U_n} = \lim_{n\to\alpha} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(n+1)(n+1)}$$

$$= \lim_{n\to\alpha} \frac{n(3+3/n)\, n(3+2/n)\, n(3+1/n)}{n(1+1/n)\, n(1+1/n)\, n(1+1/n)}$$

$$= \frac{(3+0)(3+0)(3+0)}{(1+0)(1+0)(1+0)}$$

$$= 27.$$

$$\therefore \lim_{n\to\alpha} \left(\frac{\lfloor 3n}{\lfloor n \lfloor n \lfloor n}\right)^{\frac{1}{n}} = 27.$$

Ex-31 Prove that $\lim_{n\to\alpha} n^{\frac{1}{n}} = 1$

Soln:- Let, $U_n = n$

$U_{n+1} = n+1$

$$\lim_{n\to\alpha} \frac{U_{n+1}}{U_n} = \lim_{n\to\alpha} \frac{n+1}{n}$$

$$= \lim_{n\to\alpha} \frac{n(1+1/n)}{n}$$

$$= (1+0) = 1$$

$$\therefore \lim_{n\to\alpha} U_n^{\frac{1}{n}} = 1$$

$$= \lim_{n\to\alpha} n^{\frac{1}{n}} = 1.$$

Ex-32 Find $\lim_{n\to\infty}\left(\frac{\lfloor 4n}{\lfloor 2n \lfloor 2n}\right)^{\frac{1}{n}}$.

Soln:- Let,

$$U_n = \frac{\lfloor 4n}{\lfloor 2n \lfloor 2n}$$

$$U_{n+1} = \frac{\lfloor 4n+4}{\lfloor 2n+2 \lfloor 2n+2}$$

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\frac{\lfloor 4n+4}{\lfloor 2n+2 \lfloor 2n+2}\cdot\frac{\lfloor 2n \lfloor 2n}{\lfloor 4n}$$

$$= \lim_{n\to\infty}\frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(2n+2)(2n+1)(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty}\frac{n(4+4/n)\, n(4+3/n)\, n(4+2/n)\, n(4+1/n)}{n(2+2/n)\, n(2+1/n)\, n(2+2/n)\, n(2+1/n)}$$

$$= \frac{(4+0)(4+0)(4+0)(4+0)}{(2+0)(2+0)(2+0)(2+0)}$$

$$= 16.$$

$\therefore \lim_{n\to\infty} U_n^{\frac{1}{n}} = 16$

$\Rightarrow \lim_{n\to\infty}\left(\frac{\lfloor 4n}{\lfloor 2n \lfloor 2n}\right)^{\frac{1}{n}} = 16$

Ex-33 Prove that $\lim_{n\to\infty}\frac{x^n}{n} = 0$ if $|x| \leq 1$.

Soln:- $U_n = \frac{x^n}{n}$

$U_{n+1} = \frac{x^{n+1}}{n+1}$

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}$$

$$= \lim_{n\to\infty}\frac{x \cdot n}{n(1+\frac{1}{n})}$$

$$= x \leq 1.$$

$\lim_{n\to\infty} U_n = 0 \Rightarrow \lim_{n\to\infty}\frac{x^n}{n} = 0$.

Cauchy limit theorem

If $\lim_{n\to\infty} U_n = l$, then $\frac{U_1 + U_2 + \cdots U_n}{n} \to l$, as $n \to \infty$.

Ex-34 Prove that $\lim_{n\to\infty} \frac{1+\sqrt{2}+3\sqrt{3}+4\sqrt{4}+\cdots n\sqrt{n}}{n} = 1$

Soln:- $U_n = n\sqrt{n} = n^{\frac{1}{n}}$

Let $V_n = n$

$V_{n+1} = n+1$

$\lim_{n\to\infty} \frac{V_{n+1}}{V_n} = \lim_{n\to\infty} \frac{n+1}{n}$

$= \lim_{n\to\infty} \frac{n(1+1/n)}{n}$

$= 1.$

$\therefore \lim_{n\to\infty} V_n^{\frac{1}{n}} = 1$

$\Rightarrow \lim_{n\to\infty} n^{\frac{1}{n}} = 1$

$\Rightarrow \lim_{n\to\infty} U_n = 1.$

So, by Cauchy limit theorem -

$\lim_{n\to\infty} \frac{U_1+U_2+\cdots U_n}{n} = 1.$

$\Rightarrow \lim_{n\to\infty} \cancel{\frac{1+\sqrt{2}+3\sqrt{3}+\cdots n\sqrt{n}}{n}} = 1$

$\lim_{n\to\infty} \frac{U_1+U_2+\cdots U_n}{n} = 1$

$= \lim_{n\to\infty} \frac{1+2^{1/2}+3^{1/3}+\cdots n^{1/n}}{n} = 1$

$= \lim_{n\to\infty} \frac{1+\sqrt{2}+3\sqrt{3}+\cdots n\sqrt{n}}{n} = 1$ //