RING THEORY

Prepared by

Mr. Satyajit Gayan

Assistant Professor, Silapathar College

RING:

As the preceding examples indicate, a ring is basically a set in which we have a way of adding, subtracting, multiplying, but not necessarily dividing[2] Of course, depending on the ring, the addition and multiplication may not seem like the ordinary operations we are used to. So here's the formal definition:

Definition 1.2.1. A **ring** is a set R endowed with two binary operations, usually denoted $+$ and $\cdot$, such that

- R1: R is an abelian group with respect to $+$
- R2: For any a, b, c in R, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity of $\cdot$)
- R3: For any a, b, c in R, $a \cdot (b + c) = a \cdot b + a \cdot c$ (left-distributivity)
- R3': For any a, b, c in R, $(a + b) \cdot c = a \cdot c + b \cdot c$ (right-distributivity)

Most often we will also impose some additional conditions on our rings, as follows:

- R4: There exists an element, denoted 1, which has the property that $a \cdot 1 = 1 \cdot a = a$ for all a in R (multiplicative identity)

NOTE

From now on, except in certain specific examples, if the term "ring" is used, it will mean a ring satisfying R1-R5. That is to say, unless stated otherwise, all our rings will be unital rings.

Lemma 1.2.2. *Let R be a ring, with additive and multiplicative identities 0 and 1, respectively. Then for all a, b in R,*

1. $0a = a0 = 0$;
2. $(-a)b = a(-b) = -(ab)$;
3. $(-a)(-b) = ab$;
4. $(na)b = a(nb) = n(ab)$ *for any n in $\mathbb{Z}$.*

In 4, note that n is not to be thought of as an element of R: the notation na just means $a + \cdots + a$, where there are n copies of a in the sum.

Proof. 1. Exercise

2. To show that $(-a)b = -(ab)$ is to show that the element $(-a)b$ is the additive inverse of ab; so we add them together, and hope to get zero. So $(-a)b + ab = ((-a) + a)b = (0)b = 0$ (by 1). The equality of $a(-b)$ and $-(ab)$ is similar.

3. Exercise

4. $(na)b = (a + \cdots + a)b = (ab + \cdots + ab) = n(ab) = a(b + \cdots + b) = a(nb)$

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Some Special elements in a ring:

Definition 1.3.1. Let a be an element of a ring R. We say that a is:

1. a **unit** if a has a multiplicative inverse, i.e., if there exists an element b in R such that $ab = ba = 1$; in this case, a is also said to be **invertible**, and b the **inverse** of a (and vice versa). Note also that b is a unit as well - units come in pairs. Of course, it's possible that $b = a$, i.e., an element may be its own inverse. The set of units in R is denoted $R^\times$ - the **group of units** of R;

2. a **zerodivisor** if $a \neq 0$ and there is a nonzero element b in R such that $ab = ba = 0$;

3. **nilpotent** if $a^k = 0$ for some $k \in \mathbb{N}$;

4. **idempotent** if $a^2 = a$.

Example 1.3.2. 1. In any ring 0 and 1 are (trivially) idempotent, and 0 is trivially nilpotent. 1 is always a unit ("unity is a unit")

2. In $\mathbb{Z}$, the units are ± 1, there are no zerodivisors, no nilpotent elements, and only 1 is idempotent.

3. In $\mathbb{Q}[x]$, the units are the nonzero constant polynomials, there are no zerodivisors, and no nontrivial idempotent or nilpotent elements.

4. In $M_n(\mathbb{R})$, the units are just the invertible matrices, which is just the multiplicative group $GL_n(\mathbb{R})$. There are plenty of zerodivisors: any strictly upper-triangular matrix multiplied by a strictly lower-triangular matrix is zero, so there are already lots of them. In fact, the zero-divisors are precisely the non-invertible matrices (except for 0, which never counts as a zerodivisor). This doesn't usually happen: in general, rings can contain many elements that are neither units nor zerodivisors. Nilpotents must have 0 as their only eigenvalue. Idempotents must be diagonalizable and have 0 or 1 as their only eigenvalue.

5. In $\mathbb{Z}/n\mathbb{Z}$, the units are those classes $\overline{m}$ for which $\gcd(m, n) = 1$. The zerodivisors are those for which $\gcd(m, n) \neq 1$. This is another ring in which every nonzero element is either a unit or a zerodivisor, but again do not be tempted to believe that this holds for all rings!