ϵ, δ DEFINITION OF LIMIT AND CONTINUITY OF FUNCTIONS

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Content of the Course

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1. ϵ-Neighbourhood of a point, limit point of a set.
2. Why we study Limit?
3. Limit of Functions.
4. Continuous Function.

ϵ-Neighbourhood of a point, limit point of a set

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Definition

Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then ϵ-**Neighbourhood** of a is the set $V_\epsilon = \{x \in \mathbb{R} : |x - a| < \epsilon\}$.

For $a \in \mathbb{R}$, the statement that $x \in V_\epsilon$ is equivalent to either of the statements

$$-\epsilon < x - a < \epsilon \Leftrightarrow a - \epsilon < x < a + \epsilon.$$

ϵ-Neighbourhood of a point, limit point of a set

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Definition

Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **limit point** of A if every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ such that in $|x - c| < \delta$.

Why we study Limit?

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$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

Table: Sample Table

x	f(x)	x	f(x)	x	f(x)
0	-1	1.7	1.9	2.3	1.3
1	0	1.9	0.9	2.1	1.1
-1	-2	1.99	0.99	2.01	1.01
2	??	1.999	0.999	2.001	1.001

Limit of Functions (ϵ, δ DEFINITION)

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Definition

Let $A \subseteq \mathbb{R}$ and let c be a limit point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of** f at c if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Example

1. $f : \mathbb{N} \to \mathbb{R}$, $f(x) = x + 1$.
2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x \sin \frac{1}{x}$, prove that $\lim_{x \to 0} f(x) = 0$.

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Example

Solution:
Given $|f(x) - 0| = |x \sin \frac{1}{x}|$.
$= |x||\sin \frac{1}{x}| \leq |x - o|$
Thus choosing a $\delta = \epsilon$,
we see that $|x \sin \frac{1}{x}| < \epsilon$, when $0 < |x| < \delta$
Therefore, $\lim_{x \to 0} f(x) = 0$

Continuity of Functions (ϵ, δ DEFINITION)

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Definition

Let $A \subseteq \mathbb{R}$, let function $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

- If f fails to be continuous at c, then we say that f is **discontinuous at** c.
- If $c \in A$ is a limit point of A, then by a comparison of Definitions of limit and continuity we can say that f is continuous at c if and only if $f(c) = \lim_{x \to c} f(x)$.

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Continuity of Functions (ϵ, δ DEFINITION)

Example

- $f : \mathbb{N} \to \mathbb{R}$, $f(x) = x + 1$, is a continuous function.
- if f(x)= $\begin{cases} x \sin \frac{1}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$
  then f is continuous at $x = 0$.

THANK YOU