Limit inferior and Limit Superior of SEQUENCES

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Content of Presentation

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Limit points of a Sequence.

Definition

A real number a is said to be a limit point of a sequence $< a_n >$. If for any $\epsilon > 0$, there exists $a_k \in (a-\epsilon, a+\epsilon)$, for an infinite numbers of values of k, i.e., $|a_n - a| < \epsilon$, for an infinite numbers of values of n.

Example

1. $< a_n > = < n^2 >$, $n \in \mathbb{N}$, has no limit point.
2. $< a_n > = < 2 >$, $n \in \mathbb{N}$, has the only limit point 2.
3. $< a_n > = < (-1)^n >$, $n \in \mathbb{N}$, has two limit points 1 and -1.
4. $< a_n > = < \frac{1}{n} >$, $n \in \mathbb{N}$, has 0 as a unique limit point.

Some Theorems

Definition

A sequence $< a_n >$ is said to be bounded if there exists a real number $M > 0$ such that $|a_n| \leq M$ $\forall$n $\in \mathbb{N}$.

Theorem

A convergent sequence of real numbers is bounded.

Theorem

A sequence cannot converge to more than one limit.

Some Theorems

Theorem

Bolzano-Weierstrass Theorem
Every bounded sequence has a limit point.

Theorem

A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.

Limit inferior and Limit Superior

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Definition

Let $< a_n >$ be a bounded sequence of real numbers.

1. The limit superior $\lim_{n\to\infty} sup \;\; a_n$ of $< a_n >$ is

$$\lim_{n\to\infty} sup \;\; a_n = inf_n \;\; sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

2. The limit inferior $\lim_{n\to\infty} inf \;\; a_n$ of $< a_n >$ is

$$\lim_{n\to\infty} inf \;\; a_n = sup_n \;\; inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

Limit inferior and Limit Superior

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Theorem

If $< a_n >$ is a bounded sequence, then

1. $\liminf a_n =$ *smallest limit point of* $< a_n >$, *and*
2. $\limsup a_n =$ *greatest limit point of* $< a_n >$.

Theorem

A bounded sequence $< a_n >$ is converges to a real number a if and only if

$$\liminf a_n = \limsup a_n = a$$

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Example

Consider the sequence $a_n = (1 + (-1)^n \frac{1}{n})^n$ then

1. $\liminf a_n = ?$

2. $\limsup a_n = ?$

Ans:-

$$a_n = \begin{cases} (1 + \frac{1}{n})^n, \text{if} \quad n \quad is \quad even \\ (1 - \frac{1}{n})^n, \text{if} \quad n \quad is \quad odd \end{cases}$$

Therefore the set of limit points of $< a_n >= \{e, \frac{1}{e}\}$.
So, $\liminf a_n = e$ and $\limsup a_n = \frac{1}{e}$

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THANK YOU