1. First partial derivatives

The x partial derivative

For a function of a single variable, $y = f(x)$, changing the independent variable x leads to a corresponding change in the dependent variable y. The **rate of change** of y with respect to x is given by the **derivative**, written $\frac{df}{dx}$. A similar situation occurs with functions of more than one variable. For clarity we shall concentrate on functions of just two variables.

In the relation $z = f(x, y)$ the **independent** variables are x and y and the **dependent variable** z. We have seen in Section 18.1 that as x and y vary the z-value traces out a surface. Now both of the variables x and y may change *simultaneously* inducing a change in z. However, rather than consider this general situation, to begin with we shall hold one of the independent variables **fixed**. This is equivalent to moving along a curve obtained by intersecting the surface by one of the coordinate planes.

Consider $f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$.

Suppose we keep y constant and vary x; then what is the rate of change of the function f?

Suppose we hold y at the value 3 then

$$f(x, 3) = x^3 + 6x^2 + 9 + 2x + 1 = x^3 + 6x^2 + 2x + 10$$

In effect, we now have a function of x only. If we differentiate it with respect to x we obtain the expression:

$$3x^2 + 12x + 2.$$

We say that f has been **partially differentiated** with respect to x. We denote the partial derivative of f with respect to x by $\frac{\partial f}{\partial x}$ (to be read as 'partial dee f by dee x'). In this example, when $y = 3$:

$$\frac{\partial f}{\partial x} = 3x^2 + 12x + 2.$$

In the same way if y is held at the value 4 then $f(x, 4) = x^3 + 8x^2 + 16 + 2x + 1 = x^3 + 8x^2 + 2x + 17$ and so, for this value of y

$$\frac{\partial f}{\partial x} = 3x^2 + 16x + 2.$$

Now if we return to the original formulation

$$f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$$

and treat y as a constant then the process of partial differentiation with respect to x gives

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 + 4xy + 0 + 2 + 0 \\ &= 3x^2 + 4xy + 2.\end{aligned}$$

NOTE:

The Partial Derivative of f with respect to x

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$ and is obtained by differentiating $f(x, y)$ with respect to x in the usual way but treating the y-variable as if it were a constant.

Alternative notations for $\dfrac{\partial f}{\partial x}$ are $f_x(x, y)$ or f_x or $\dfrac{\partial z}{\partial x}$.

NOTE:

The Partial Derivative of f with respect to y

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to y is denoted by $\dfrac{\partial f}{\partial y}$ and is obtained by differentiating $f(x, y)$ with respect to y in the usual way but treating the x-variable as if it were a constant.

Alternative notations for $\dfrac{\partial f}{\partial y}$ are $f_y(x, y)$ or f_y or $\dfrac{\partial z}{\partial y}$.

EX:

Given $f(x, y) = 3x^2 + 2y^2 + xy^3$ find $f_x(1, -2)$ and $f_y(-1, -1)$.

First find expressions for $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$:

Your solution

$\dfrac{\partial f}{\partial x} =$ $\qquad\qquad$ $\dfrac{\partial f}{\partial y} =$

Answer

$\dfrac{\partial f}{\partial x} = 6x + y^3, \quad \dfrac{\partial f}{\partial y} = 4y + 3xy^2$

Functions of several variables

As we have seen, a function of two variables $f(x,y)$ has two partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. In an exactly analogous way a function of three variables $f(x,y,u)$ has three partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial u}$, and so on for functions of more than three variables. Each partial derivative is obtained in the same way as stated in Key Point 3:

NOTE:

The Partial Derivatives of $f(x,y,u,v,w,\ldots)$

For a function of several variables $z = f(x,y,u,v,w,\ldots)$ the partial derivative of f with respect to v (say) is denoted by $\frac{\partial f}{\partial v}$ and is obtained by differentiating $f(x,y,u,v,w,\ldots)$ with respect to v in the usual way but treating all the other variables as if they were constants.

Alternative notations for $\frac{\partial f}{\partial v}$ when $z = f(x,y,u,v,w,\ldots)$ are $f_v(x,y,u,v,w\ldots)$ and f_v and $\frac{\partial z}{\partial v}$.

EX:

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$ for $f(x,y,u,v) = x^2 + xy^2 + y^2u^3 - 7uv^4$

Your solution

$\frac{\partial f}{\partial x} =$ $\qquad$ $\frac{\partial f}{\partial u} =$

Answer

$\frac{\partial f}{\partial x} = 2x + y^2 + 0 + 0 = 2x + y^2$; $\quad \frac{\partial f}{\partial u} = 0 + 0 + y^2 \times 3u^2 - 7v^4 = 3y^2u^2 - 7v^4$.

2. Second partial derivatives

Performing two successive partial differentiations of $f(x,y)$ with respect to x (holding y constant) is denoted by $\frac{\partial^2 f}{\partial x^2}$ (or $f_{xx}(x,y)$) and is defined by

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

For functions of two or more variables as well as $\frac{\partial^2 f}{\partial x^2}$ other second-order partial derivatives can be obtained. Most obvious is the second derivative of $f(x,y)$ with respect to y is denoted by $\frac{\partial^2 f}{\partial y^2}$ (or $f_{yy}(x,y)$) which is defined as:

$$\frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

EX:

Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ for $f(x,y) = x^3 + x^2y^2 + 2y^3 + 2x + y$.

Solution

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 + 0 + 2 + 0 = 3x^2 + 2xy^2 + 2$$

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x + 2y^2 + 0 = 6x + 2y^2.$$

$$\frac{\partial f}{\partial y} = 0 + x^2 \times 2y + 6y^2 + 0 + 1 = 2x^2y + 6y^2 + 1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2x^2 + 12y.$$

EX:

Find $f_{xx}(-1,1)$ and $f_{yy}(2,-2)$ for $f(x,y) = x^3 + x^2y^2 + 2y^3 + 2x + y$.

Solution

$$f_{xx}(-1,1) = 6 \times (-1) + 2 \times (-1)^2 = -4.$$

$$f_{yy}(2,-2) = 2 \times (2)^2 + 12 \times (-2) = -16$$

3.

Mixed second derivatives

It is possible to carry out a partial differentiation of $f(x,y)$ with respect to x followed by a partial differentiation with respect to y (or vice-versa). The results are examples of **mixed derivatives**. We must be careful with the notation here.

We use $\dfrac{\partial^2 f}{\partial x \partial y}$ to mean 'differentiate first with respect to y and then with respect to x' and we use $\dfrac{\partial^2 f}{\partial y \partial x}$ to mean 'differentiate first with respect to x and then with respect to y':

$$\text{i.e.} \qquad \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

(This explains why the order is opposite of what we expect - the derivative 'operates on the left'.)

EX:

For $f(x,y) = x^3 + 2x^2y^2 + y^3$ find $\frac{\partial^2 f}{\partial x \partial y}$.

Solution

$$\frac{\partial f}{\partial y} = 4x^2y + 3y^2; \quad \frac{\partial^2 f}{\partial x \partial y} = 8xy$$

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