NOTES ON GRAPH THEORY

A simple graph is **a graph that does not have more than one edge between any two vertices and no edge starts and ends at the same vertex**. In other words a simple graph is a graph without loops and multiple edges. Adjacent Vertices. Two vertices are said to be adjacent if there is an edge (arc) connecting them.

- **Adjacent vertices:**

Given two vertices u and v, if $uv \in E$, then u and v are said to be *adjacent*. In this case, u and v are said to be the *end vertices* of the edge uv. If $uv \notin E$, then u and v are *nonadjacent*. Furthermore, if an edge e has a vertex v as an end vertex, we say that v is *incident* with e.

Question1:

Solution:

Let G be a graph with $'e'$ edges and $'n'$ vertices $v_1, v_2, v_3, \ldots, v_n$. Since each edge is incident on two vertices, it contributes 2to the sum of degree of vertices in graph G. Thus the sum of degrees of all vertices in G is twice the number of edges in G. Hence,

$$\sum_{i=1}^{n} \text{degree}(v_i) = 2e.$$

Let the degrees of first r vertices be even and the remaining $(n-r)$ vertices have odd degrees,then clearly, $\sum_{i=1}^{r} \text{degree}(v_i) = even$.Since,

$$\sum_{i=1}^{n} \text{degree}(v_i) = \sum_{i=1}^{r} \text{degree}(v_i) + \sum_{i=r+1}^{n} \text{degree}(v_i)$$

$\implies \sum_{i=1}^{n} \text{degree}(v_i) - \sum_{i=1}^{r} \text{degree}(v_i) = \sum_{i=r+1}^{n} \text{degree}(v_i)$

$\implies \sum_{i=r+1}^{n} \text{degree}(v_i)$ is even.$(WHY?)$

But, the for $i = r+1, r+2, \ldots, n$ each $d(v_i)$ is odd. So, the number of terms in $\sum_{i=r+1}^{n} \text{degree}(v_i)$ must be even.

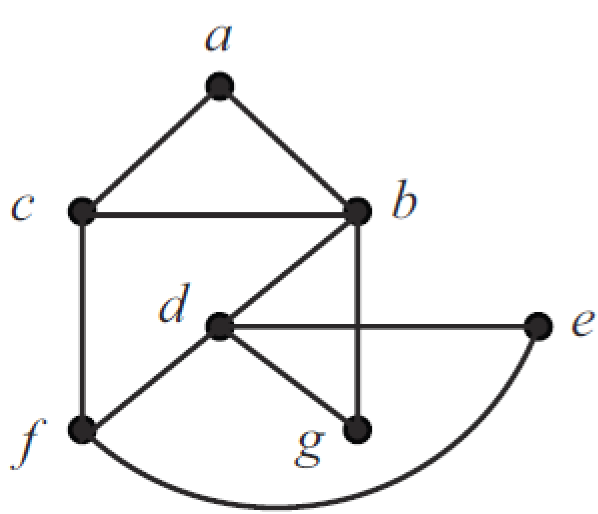
In lucid words, $(n-r)$ is even.

Hence the result.

- **Walk, Path, and Trail:**

A *walk* in a graph is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, k-1$. Such a walk is sometimes called a v_1–v_k *walk*, and v_1 and v_k are the *end vertices* of the walk. If the vertices in a walk are distinct, then the walk is called a *path*. If the edges in a walk are distinct, then the walk is called a *trail*. In this way, every path is a trail, but not every trail is a path. Got it?

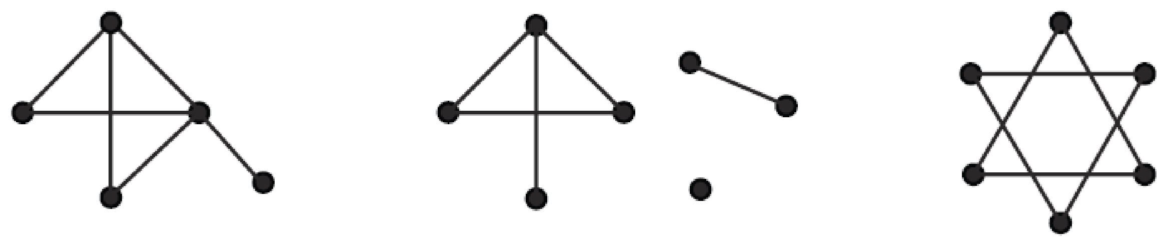
A *closed path*, or *cycle*, is a path $v_1, \ldots, v_k$ (where $k \geq 3$) together with the edge $v_k v_1$. Similarly, a trail that begins and ends at the same vertex is called a *closed trail*, or *circuit*. The *length* of a walk (or path, or trail, or cycle, or circuit) is its number of edges, counting repetitions.



Once again, let's illustrate these definitions with an example. In the graph of Figure 1.7, a, c, f, c, b, d is a walk of length 5. The sequence b, a, c, b, d represents a trail of length 4, and the sequence d, g, b, a, c, f, e represents a path of length 6. Also, g, d, b, c, a, b, g is a circuit, while e, d, b, a, c, f, e is a cycle. In general, it is possible for a walk, trail, or path to have length 0, but the least possible length of a circuit or cycle is 3.

- **Connected Graph**

A graph is *connected* if every pair of vertices can be joined by a path. Informally, if one can pick up an entire graph by grabbing just one vertex, then the

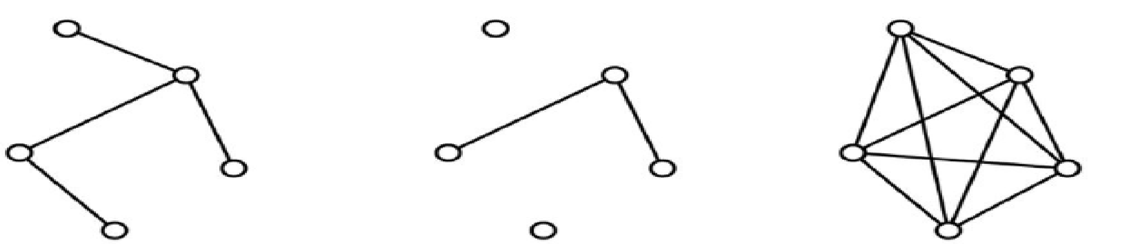


G_1 G_2 G_3

FIGURE 1.9. Connected and disconnected graphs.

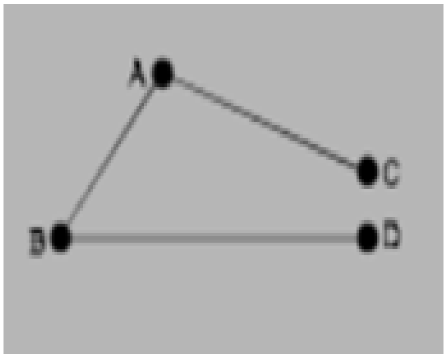
graph is connected. In Figure 1.9, G_1 is connected, and both G_2 and G_3 are not connected (or *disconnected*). Each maximal connected piece of a graph is called a *connected component*. In Figure 1.9, G_1 has one component, G_2 has three components, and G_3 has two components.

- A ***connected graph*** has a path between each pair of distinct vertices.
- A ***complete graph*** has an edge between each pair of distinct vertices.
  - A complete graph is also a connected graph. But a connected graph may not be a complete graph.

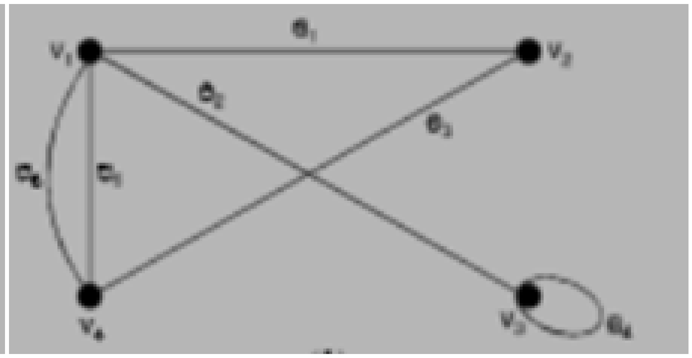


(a) **connected** (b) **disconnected** (c) **complete**

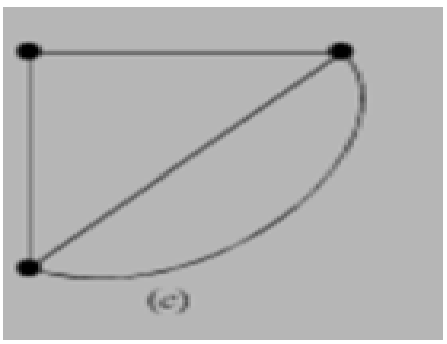
Question2: Find which of the following graphs are connected and which are not connected.



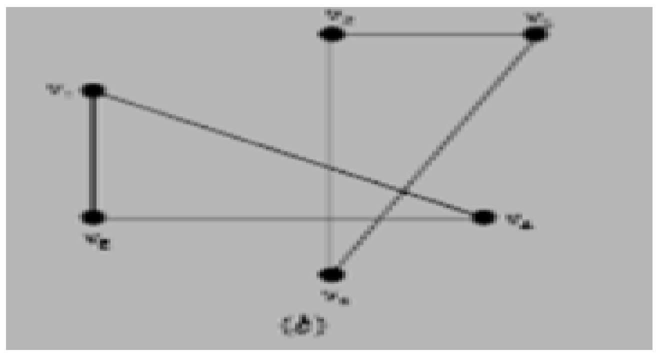
(a)



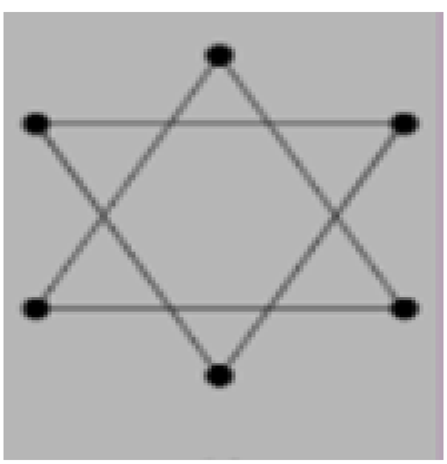
(b)



(c)



(d)



(e)