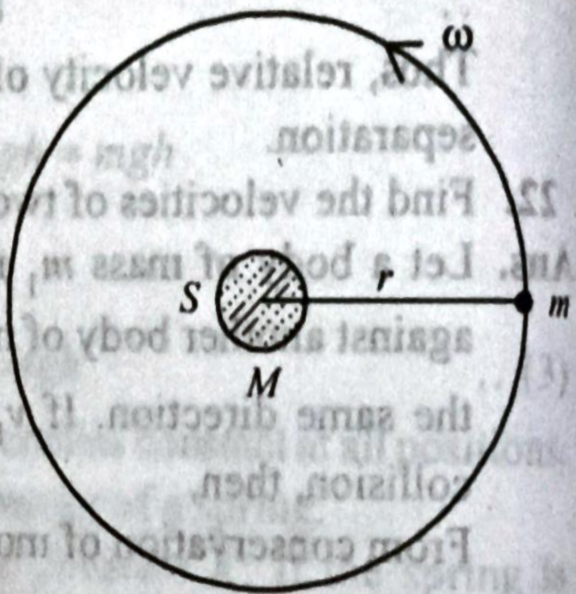
Deduce Kepler's third law from Newton's law of gravitation.

To a first approximation, the orbit of the planet can be considered to be circular.

Let us consider a planet of mass *m* moving around the sun of mass *M* in a circular orbit of radius *r* with a uniform angular velocity ω. Then, for equilibrium,



$$G\frac{Mm}{r^2} = mrw^2$$

$$\Rightarrow \quad GM = r^3 w^2$$

$$\Rightarrow \quad GM = r^3 . \frac{u\pi^2}{T^2}$$

$$\Rightarrow \quad T^2 = \frac{u\pi^2}{GM} . r^3$$

As, $\frac{u\pi^2}{GM}$ = constant,

So, $O T^2 \propto r^3$.

This is Kepler's third law.

. Deduce Newton's law of gravitation from Kepler's law.

. Let a planet of mass *m* be revolving around the sun of mass *M* in a nearly circular orbit of radius *r* with a uniform angular velocity ω. If *T* is the period of revolution of the planet, then

$$\omega = \frac{2\pi}{T}$$

Now, the centripetal force acting on the planet is,

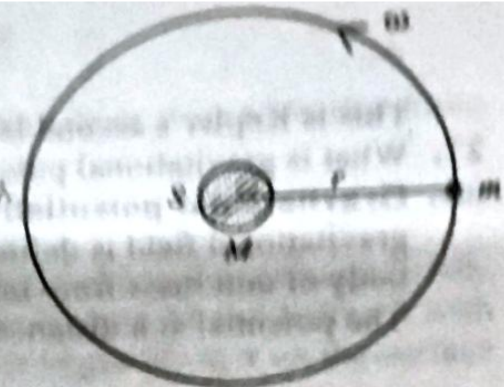
$$F = mrw^2 = mr\frac{4\pi^2}{T^2} \qquad ...(1)$$

According to Kepler's 3rd law,

$$T^2 \propto r^3$$

∴ $$T^2 = kr^3$$

Where K is constant of proportionality.



∴ $$F = mr\frac{4\pi^2}{kr^3} = \frac{4\pi^2 m}{kr^2} \quad ...(2)$$

As $\frac{4\pi^2 m}{K}$ = Constant, so, $F \propto \frac{1}{r^2}$

This centripetal force is provided by the gravitational attractive exerted by the planet on sun. As gravitational force is a mutual force and as it depends on the mass of the planet, so it must also depend on the mass of sun M.

∴ $$\frac{4\pi^2}{K} \propto M$$

i.e., $$\frac{4\pi^2}{K} = GM$$

Where G is the constant of proportionality. Putting this value in eqn. (2), we get,

$$F = G\frac{Mm}{r^2}$$

This is Newton's law of gravitation.

5. Deduce Kepler's second law.

Ans. If $\frac{dA}{dt}$ is the area swept out by the radius vector of a particle per unit time *i.e.*, if $\frac{dA}{dt}$ is the areal velocity of a particle, then the angular momentum of the particle of mass m is given by,

$$L = 2m\frac{dA}{dt} \quad ...(1)$$

Let a planet of mass m be revolving around the sun in a circular orbit of radius r. The force of attraction $\vec{F}$ on the planet due to the sun acts along the line joining the centre of the sun and the planet and is directed towards the sun. So, the angle between $\vec{F}$ and $\vec{r}$ is 180°.

Torque on the planet $= \tau = \vec{r} \times \vec{F} = rF\sin\theta$

∴ $$\tau = rF\sin 180° = 0$$

Now, $$\tau = \frac{dL}{dt}$$

∴ $$\frac{dL}{dt} = 0$$

i.e., L = constant.

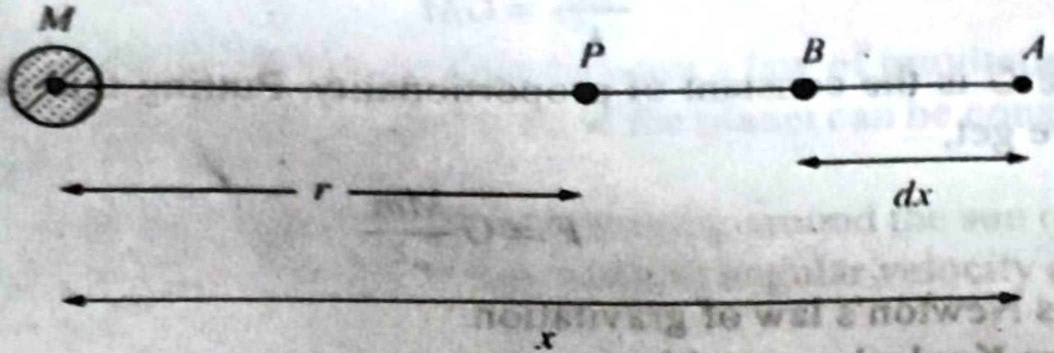
Hence from eqn. (1),

$$2m\frac{dA}{dt} = \text{Constant}$$

Find the gravitational potential in the gravitational field of earth.

Let M be the mass and R be the radius of earth. The mass of the earth can be supposed to be concentrated at its centre.

Let us find the potential at P at a distance r from the centre of the earth $(r > R)$.



The force on a unit mass at A at a distance x from the centre of the earth is,

$$F = G\frac{M}{x^2}.$$

The work done in moving the unit mass through a small distance dx from A to B is given by,

$$dW = Fdx = G\frac{M}{x^2}dx.$$

Hence the total work done in bringing the unit mass from infinity to P, *i.e.*, the potential at P is given by,

$$V = W = \int_{\infty}^{r} \frac{GM}{x^2}dx$$

$$= \int_{\infty}^{r} GMx^{-2}dx$$

$$= -GM\left[\frac{1}{x}\right]_{\infty}^{r}$$

$$= -GM\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$= -\frac{GM}{r}$$

$$\therefore \quad V = -\frac{GM}{r}.$$

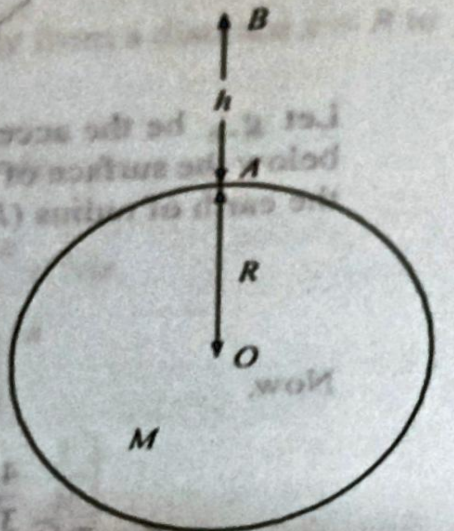
The negative sign means that gravitational force is an attractive force.
Potential on the surface of the earth is

$$V_s = -G\frac{M}{R}.$$

Find the value of acceleration due to gravity above the surface of earth.
Let us consider the earth to be a sphere of mass M and radius R with centre at O. If g_0 is the acceleration due to gravity at A on the surface of the earth, then,

$$g_0 = G\frac{M}{R^2} \quad ...(1)$$

If g_h is the acceleration due to gravity at the point B at a height h above the surface of the earth, then,



$$g_h = G\frac{M}{(R+h)^2} \quad ...(2)$$

From (1) and (2),

$$\frac{g_h}{g_0} = \frac{R^2}{(R+h)^2}$$

$$\Rightarrow \frac{g_h}{g_0} = \frac{R^2}{R^2\left(1+\frac{h}{R}\right)^2}$$

$$\Rightarrow g_h = \frac{g_0}{\left(1+\frac{h}{R}\right)^2} \quad ...(3)$$

$\therefore g_h < g_0$.

So, acceleration due to gravity decreases with altitude.

From (1), $g_h = g_0\left(1+\frac{h}{R}\right)^{-2}$

If $h << R$, then,

$$g_h = g_0\left(1-\frac{h}{R}\right).$$

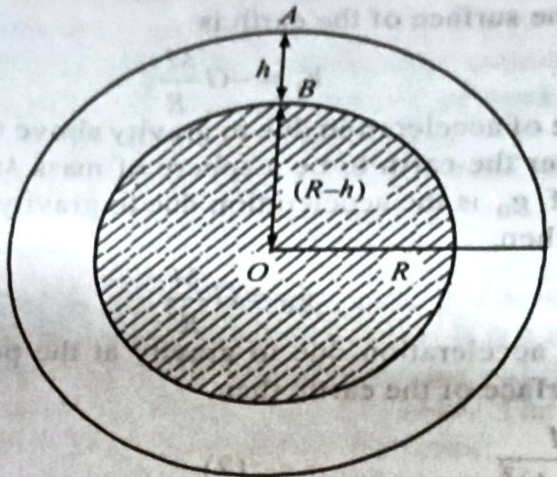
Find the value of acceleration due to gravity below the surface of earth.
Let us consider the earth to be a homogeneous sphere of mass M and radius R with centre at O. If g is the acceleration due to gravity at the point A on the surface of the earth, then, $g = G\frac{M}{R^2}$.

If ρ is the densidity of earth, then

$$M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore \quad g = \frac{G\frac{4}{3}\pi R^3 \rho}{R^2}$$

i.e., $$g = \frac{4}{3}\pi G R \rho$$



Let g_{-h} be the acceleration due to gravity at the point B at a depth h below the surface of earth. The body will be attracted by the portion of the earth of radius $(R-h)$ and mass M'.

$$\therefore \quad g_{-h} = G\frac{M'}{(R-h)^2}$$

Now, $$M' = \frac{4}{3}\pi(R-h)^3\rho$$

$$\therefore \quad g_{-h} = G\frac{\frac{4}{3}\pi(R-h)^3\rho}{R-h} = \frac{4}{3}\pi G(R-h)\rho \quad \ldots(2)$$

From (1) and (2)

$$\frac{g_{-h}}{g} = \frac{\frac{4}{3}\pi G(R-h)\rho}{\frac{4}{3}\pi G R\rho} = \frac{R-h}{R} = 1 - \frac{h}{R}$$

$$\therefore \quad g_{-h} = g\left(1 - \frac{h}{R}\right)$$

i.e., $$g_{-h} < g$$

Thus, acceleration due to gravity decreases with depth.

If $h = R$, *i.e.,* at the centre of the earth,

$$g_{-h} = g(1-1) = 0$$

So, the acceleration due to gravity at the centre of the earth is zero. Hence a body becomes weightless at the centre of the earth.

State the laws of falling bodies.

The followings are the laws of falling bodies.

(*i*) All bodies, falling freely from rest in vacuum, fall with equal rapidity.

(*ii*) The velocity acquired by a body, falling freely from rest in vacuum, is directly proportional to the time of fall,

i.e., $v \propto t$.

(*iii*) The distance through which a body falls freely from rest in vacuum is directly proportional to the square of the time of fall.

$$i.e., h \propto t^2$$

Find the work done in taking a body from the surface of the earth to a height equal to the radius of the earth.

The force of gravity on a body of mass m at a distance x from the centre of the earth of mass M and radius R is given by,

$$F = G\frac{Mm}{x^2}$$

where, G = universal gravitational constant.

The work done in taking the body through a small distance dx is,

$$dW = -F\,dx = -G\frac{Mm}{x^2}dx$$

Hence, the total work done in taking the body from a distance $x = R$ to a distance $x = 2R$ is,

$$W = -\int_R^{2R} G\frac{Mm}{x^2}dx$$

$$= -G\,Mm\int_R^{2R} x^{-2}dx$$

$$= G\,Mm\left[\frac{1}{x}\right]^{2R}$$

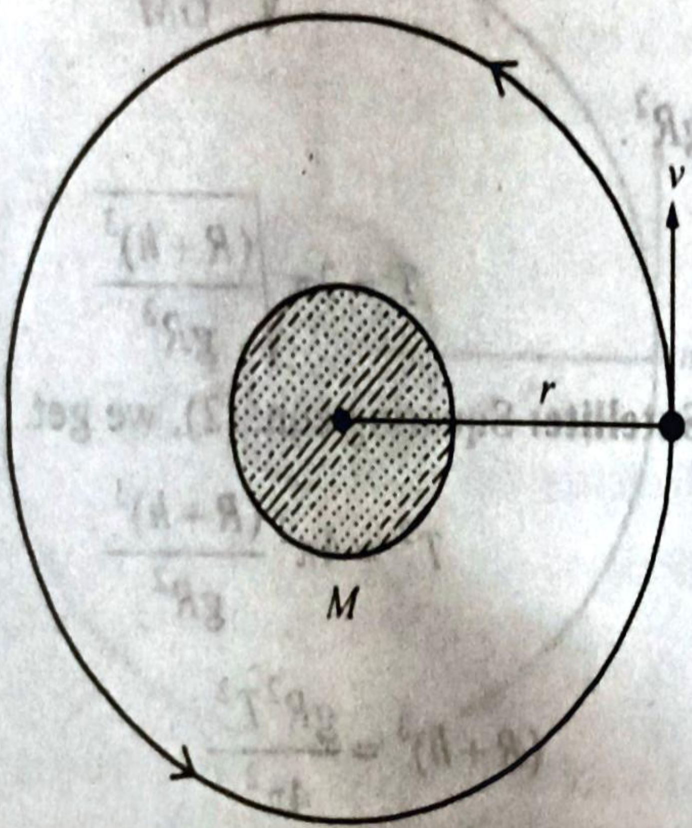
$$= G\,Mm\left[\frac{1}{2R} - \frac{1}{R}\right]$$

$$= -G\frac{Mm}{2R}$$

5. Find the energy of a satellite revolving around the earth.

Ans. Let a satellite of mass m be revolving around the earth of mass M with an orbital velocity v in a circular orbit of radius r. At any instant, the satellite possesses two types of energy.

(*i*) **Kinetic energy:** It is the energy possessed by the satellite due to its motion. Kinetic energy is given by,



$$T = \frac{1}{2}mv^2$$

Again $$\frac{mv^2}{r} = G\frac{Mm}{r}$$

i.e., $$mv^2 = G\frac{Mm}{r}$$

$\therefore$ $$T = \frac{1}{2}\frac{G\,Mm}{r} \qquad \text{...(1)}$$

(*ii*) **Potential energy:** It is the energy possessed by the satellite due to its position in the orbit.

Potential energy $$= U = -\frac{G\,Mm}{r} \qquad \text{...(2)}$$

$\therefore$ Total energy $$= E = T + U$$

$$= \frac{1}{2}\frac{G\,Mm}{r} - \frac{G\,Mm}{r}$$

$$= -\frac{1}{2}\frac{G\,Mm}{r}$$

The negative sign means that the satellite is bound to the planet.