LAWS OF MOTION
(Circular Motion)

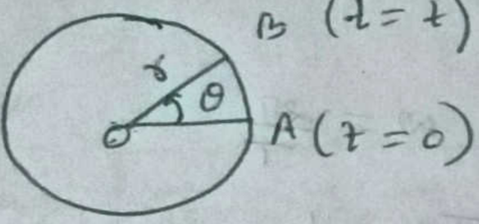
Important terms in circular motion :

1) **Angular displacement (θ)**

The angle described by the particle about the axis of rotation (or centre O) in a given time is called the angular displacement (θ).

$\angle AOB = \theta$.

Unit → Radian

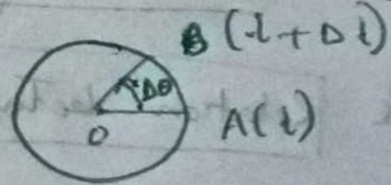


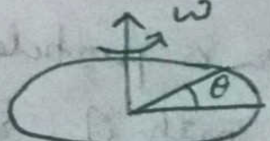
2) **Angular speed (ω)**

The ratio of angular displacement to the time taken is called angular speed. Denoted by ω (omega)

Unit → radian/sec

$$\boxed{\omega = \frac{\theta}{t}}$$





3) **Time-period (T)**

Time taken by a particle to complete one rotation is called time-period (T)

$$\boxed{\omega = \frac{2\pi}{T} \text{ rad/s}}$$

$\left[\because \theta = 2\pi \text{ when } t = T \right]$

4) <u>Frequency</u> (ν)

The number of rotations completed by a particle in one sec is called frequency.

$$\boxed{\nu = \frac{1}{T}}$$

<u>Unit is Hertz (Hz) or sec^{-1}.</u>

* **Relation between ω and ν**

We know,

$$\omega = \frac{2\pi}{T}$$

$$\because \quad T = \frac{1}{\nu}$$

$$\therefore \quad \boxed{\omega = 2\pi\nu}$$

<u>Direction of ω</u> $\rightarrow$ Along the axis of rotation

* **Relation between linear speed and angular speed**

- Let us consider a particle moving in a circular path of radius r. Let it travel linear distance 'Δs' in time 'Δt' with linear speed v and angular speed 'ω'

Let $\Delta\theta$ be the angular displacement in time Δt

then

$$\omega = \frac{\Delta\theta}{\Delta t}.$$

$$v = \frac{\Delta s}{\Delta t}.$$

And from figure

$$\Delta s = r\Delta\theta.$$

$$\therefore \quad v = r\left(\frac{\Delta\theta}{\Delta t}\right)$$

$$v = r\omega$$

In vector form, $\boxed{\vec{v} = \vec{r} \times \vec{\omega}}$.

Angular acceleration (α)

Rate of change of angular velocity of a particle.

$$\omega = \frac{d\theta}{dt}.$$

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right)$$

$$\boxed{\alpha = \frac{d^2\theta}{dt^2}}$$

* Relation between linear acceleration (a) and angular acceleration (α)

we have,

linear acceleration, $a = \frac{dv}{dt}$

But $v = r\omega$.

$\Rightarrow \quad a = \frac{d}{dt}(r\omega)$

$= r\frac{d\omega}{dt}$

$= r\alpha \quad \left[\because \alpha = \frac{d\omega}{dt}\right]$

In vector form $\boxed{\vec{a} = \vec{r} \times \vec{\alpha}}$

For uniform / constant angular velocity

$\frac{d\omega}{dt} = 0.$

$\therefore \quad \boxed{\alpha = 0}$

i.e $\omega =$ constant $\longrightarrow \frac{d\omega}{dt} = 0$ or $\alpha = 0$

Centripetal acceleration

(for Uniform circular motion)

or

Motion with constant speed.

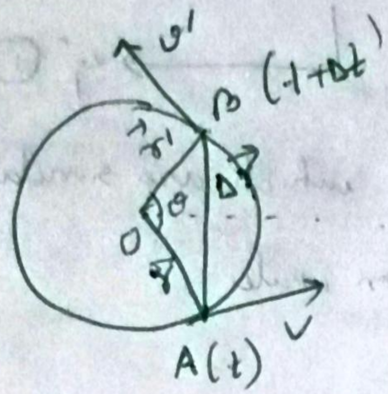


fig. a

$r + \Delta r = r'$

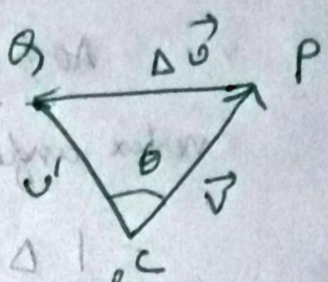


fig b

$\vec{v} + \Delta\vec{v} = v'$

(triangle law of vector addition)

Let a particle moves in a circle of radius 'r'.

Let 'v' be the velocity at point A at time 't'

v' " " " " B " " 't+Δt'.

For uniform circular motion, $v = v' = v$ (say)

To find the change in velocity Δv, we draw figure (b) where

$$\vec{PQ} = \vec{CQ} - \vec{CP} = \vec{v}' - \vec{v}$$

$$\boxed{\Delta\vec{v} = v' - v}$$

Direction of $\Delta\vec{v}$ ⟶ Towards centre of circle

Average acceleration of the particle is

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} \quad [\text{towards centre}]$$

⟶ eqⁿ ①

From triangles.

AOB and POQ which are similar with vertex angle θ, We can write

$$\frac{|\Delta\vec{v}|}{|\Delta\vec{r}|} = \frac{|\vec{v}|}{|\vec{r}|}$$

$$\Rightarrow \Delta\vec{v} = \frac{v}{r}\,\Delta\vec{r}$$

∴ equation ① implies

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v}{r}\,\frac{|\Delta\vec{r}|}{\Delta t} \longrightarrow ②$$

when $\Delta t \to 0$, Arc $AB \approx |\Delta\vec{r}|$

i.e. $\boxed{\lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = v} \longrightarrow ③$

∴ Instantaneous acceleration

$$a = \lim_{\Delta t \to 0} |\vec{a}_{av}| = \lim_{\Delta t \to 0} \frac{v}{r}\,\frac{|\Delta\vec{r}|}{\Delta t}$$

$$\Rightarrow a = \frac{v}{r} \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta t} = \frac{v}{r} \times v \quad \left[\text{from eq}^n \text{ (3)}\right]$$

$$\Rightarrow \boxed{a = \frac{v^2}{r}} \longrightarrow (4)$$

Direction of a is towards the centre, so it is called as centripetal acceleration.

Again, we know

$$v = r\omega.$$

$$\therefore a = \frac{v^2}{r} = \frac{(r\omega)^2}{r}$$

$$= r\omega^2.$$

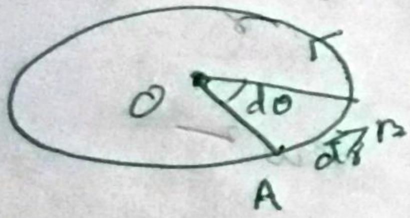
$$= r(2\pi\nu)^2 \quad [\because \omega = 2\pi\nu]$$

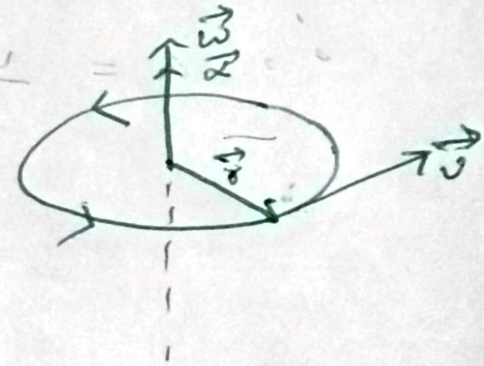
$$\boxed{a = r\,4\pi^2\nu^2}$$

Equations of motion in circular/linear motion

Linear	Circular
① $s = ut$	$\theta = \omega_0 t$
② $s = ut + \frac{1}{2} at^2$	$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$
③ $v = u + at$	$\omega = \omega_0 + \alpha t$
④ $v^2 - u^2 = 2as$	$\omega^2 - \omega_0^2 = 2 \alpha \theta$

Non-uniform circular motion





Let us consider a particle moves in a circular path of radius r with angular velocity $\vec{\omega}$ in anticlockwise direction. From A to B, let linear displacement is $\vec{dr}$ and angular displacement is $\vec{d\theta}$

$$\vec{dr} = \vec{d\theta} \times r$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}$$

$$\Rightarrow \boxed{v = \vec{\omega} \times \vec{r}} \longrightarrow (1)$$

differentiating with respect to time.

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

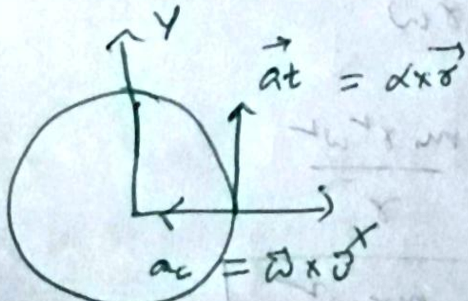
$$= \frac{d\omega}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$\Rightarrow \boxed{a = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}}$$

Here

$\vec{\alpha} \times \vec{r} = a_t$ [tangential]

$\omega \times \vec{v} = a_c$ [centripetal]



$$\boxed{a = a_t + a_c}$$

Dynamics of circular motion

Centripetal force:

The force which deviates a body from its linear path to move along a circular path is called centripetal force. It is directed towards the centre of the circular path.

We know.

$$a = \frac{v^2}{r} \quad [\text{in } eq^n\ 4]$$

Again,

$$F = ma.$$

$$\Rightarrow \boxed{F = \frac{mv^2}{r}}$$

If $v = r\omega$.

$$F = \frac{m r^2 \omega^2}{r}$$

$$\boxed{F = mr\omega^2}$$

Again $\omega = \frac{2\pi}{T}$

$$\therefore F = mr\left(\frac{2\pi}{T}\right)^2$$

$$\boxed{F = \frac{4\pi^2 mr}{T^2}}$$

Again, $\frac{1}{T} = \nu$

$$\therefore \boxed{F = 4\pi^2 m r \nu^2}$$

Example

(1) Gravitational force of attraction between Sun and earth

(2) Electrostatic force of attraction between electron and nucleus

* Centrifugal force

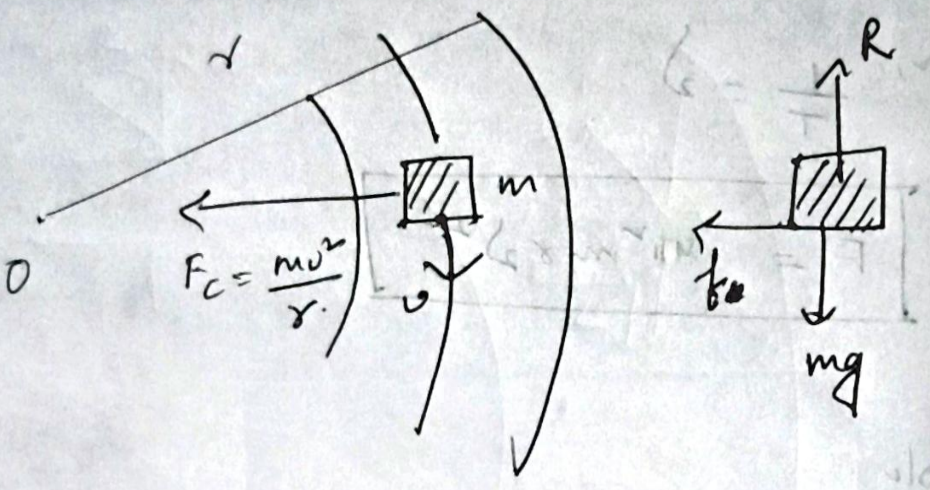
The force that tends to move the bodies / particles of a rotating frame away from the axis of rotation is called centrifugal force.

$$\boxed{F_{centrifugal} = mv^2/r}$$

* Centrifugal force is a fictitious/pseudo force that comes into play in an accelerated frame of reference.

Examples of Circular motion

① Car on a level circular road.



Let a car moves in a flat horizontal circular road of radius r

Forces acting on the car are:

① $W = mg$ ↓

② R (normal reaction) ↑.

③ Friction force 'f' between tyres of car and road.

Here $R = mg$ [∵ car is not moving in vertical direction]

In a circular path

$$F_c = \frac{mv^2}{r}$$

Here, $f = \mu R = \mu mg$

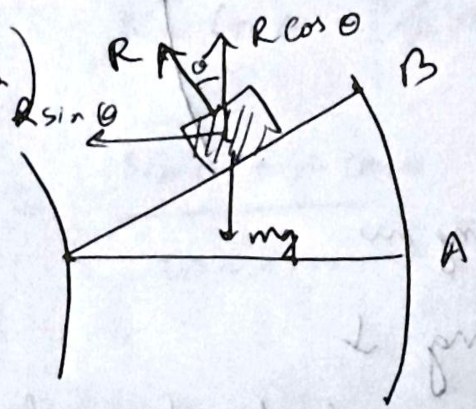
& $f = F_c$

$$\Rightarrow \frac{mv^2}{r} = \mu mg$$

$$\Rightarrow \boxed{v = \sqrt{\mu r g}}$$ ✓ Important

② Banked circular road.

(neglecting friction)



Here forces acting are —

① $w = mg \downarrow$

② R perpendicular to the surface of road ↖

Resolving into $R\sin\theta$ & $R\cos\theta$

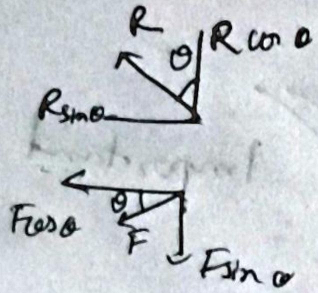
we have $R\sin\theta = \frac{mv^2}{r}$, $R\cos\theta = mg$

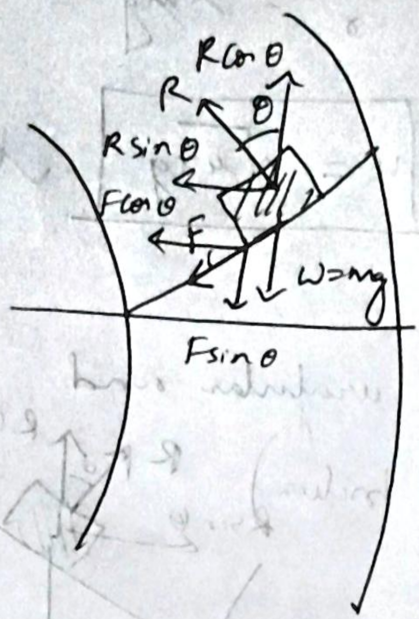
$$\tan\theta = \frac{v^2}{rg}$$

$$\Rightarrow v^2 = \tan\theta \, rg$$

$$\Rightarrow \boxed{v = \sqrt{rg\tan\theta}}$$

(3) Banked circular road (considering friction)





Forces acting on

(1) $W = mg$ ↓

(2) R perpendicular to surface

(3) force of friction F bet^n tyres & road.

For equilibrium.

$$mg + F\sin\theta = R\cos\theta$$

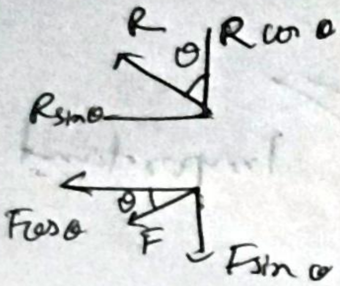
$$mg = R\cos\theta - F\sin\theta \longrightarrow (1)$$

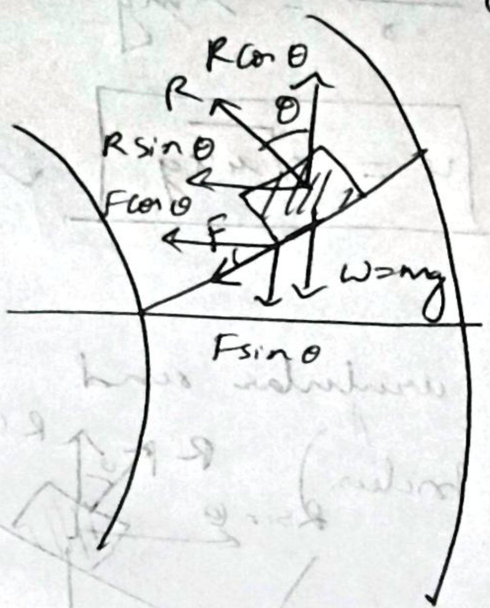
$$\tan\theta = \frac{v^2}{rg}$$

$$\Rightarrow v^2 = \tan\theta \, rg$$

$$\Rightarrow \boxed{v = \sqrt{rg\tan\theta}}$$

(3) Banked circular road (considering friction)





Forces acting are

(1) $W = mg \downarrow$

(2) R perpendicular to surface

(3) force of friction F bet^n tyres & road.

For equillibrium.

$$mg + F\sin\theta = R\cos\theta$$

$$mg = R\cos\theta - F\sin\theta \longrightarrow (1)$$

$(R\sin\theta + F\cos\theta)$ towards centre of the road

which provides centripetal force mv^2/r

$$\frac{mv^2}{r} = F\cos\theta + R\sin\theta \longrightarrow (2)$$

(2) ÷ (1)

$$\Rightarrow \frac{\frac{mv^2}{r}}{mg} = \frac{R\sin\theta + F\cos\theta}{R\cos\theta - F\sin\theta}$$

$$\Rightarrow \frac{v^2}{rg} = \frac{\cancel{R}[\sin\theta + F/R\cos\theta]}{\cancel{R}[\cos\theta - F/R\sin\theta]}$$

$$\Rightarrow \frac{v^2}{rg} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} \quad \left[\because \mu R = F \text{ or } \mu = F/R\right]$$

$$\Rightarrow \frac{v^2}{rg} = \frac{\tan\theta + \mu}{1 - \mu\tan\theta}$$ [dividing by $\cos\theta$ on numerator & denominator]

$$\Rightarrow \boxed{v = \left[\frac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}\right]^{1/2}}$$