Class Notes 4th sem C9(unit 1)

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Partitions of a closed interval

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Definition

Suppose $I = [a, b]$ is a closed and bounded interval then by a partition of I means a finite set of real numbers $P = \{x_0, x_1, x_2, \cdots, x_n\}$ having the property that $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$.

1. $I_1 = [x_0, x_1], I_2 = [x_1, x_2], \cdots I_i = [x_{i-1}, x_i], \cdots, I_n = [x_{n-1}, x_n]$ are the sub-intervals of $[a, b]$.

2. We shall use the symbol $\Delta x_i = x_{i-1} - x_i$ to denote the ith sub-interval.

3. For any partition P the length of the largest sub-interval is called norm or mesh of the partition and is denoted as $\mu(P)$.

Riemann sum

Definition

Suppose $f(x)$ is a bounded function defined in a closed interval $[a,b]$ and suppose $P = \{x_0, x_1, x_2, \cdots, x_n\}$ is a partition of $[a,b]$. Let $M_i = Sup\{f(x) : x_{i-1} \leq x \leq x_i\}$,
$m_i = Inf\{f(x) : x_{i-1} \leq x \leq x_i\}$.
Therefore $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$, is called the Lower R-sum of f on $[a,b]$ w.r.t partition P.
$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ is called the Upper R-sum of f on $[a,b]$ w.r.t partition P.

Lemma

Suppose $f(x)$ is a bounded function defined in a closed interval $[a,b]$ and suppose $P = \{x_0, x_1, x_2, \cdots, x_n\}$ is a partition of $[a,b]$. Then

$$M(b-a) \geq U(P,f) \geq L(P,f) \geq m(b-a)$$

Darboux's condition of integrability

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Definition

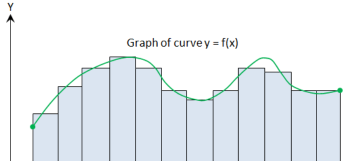
When the two integrals $\int_{\underline{a}}^{b} f dx$ and $\int_{a}^{\bar{b}} f dx$ are equal, i.e.,

$$\int_{\underline{a}}^{b} f dx = \int_{a}^{\bar{b}} f dx = \int_{a}^{b} f dx,$$

where $\int_{\underline{a}}^{b} f dx = inf\{U(P, f) : P \;\; is \;\; any \;\; partition \;\; of \;\; [a, b]\}$

and $\int_{a}^{\bar{b}} f dx = Sup\{L(P, f) : P \;\; is \;\; any \;\; partition \;\; of \;\; [a, b]\}$.

Then we say that f is R-integrable or simply integrable over $[a, b]$ and the common value of this integral is called the Riemann integral or Darboux's integral.



Examples

Example

If $f(x)$ be defined on $[0, 1]$ as

$$f(x) = \begin{cases} 1, \text{if } \ x \ \ is \ \ rational \\ -1, \text{if } \ x \ \ is \ \ irrational \end{cases} \tag{1}$$

Now, we have to show that $f(x)$ is an example of bounded function which is not R-integrable over $[0, 1]$.
Let $P = \{x_0 = 0, x_1, x_2, \cdots, x_n = 1\}$ be any partition of $[0, 1]$.
Let for any sub-interval $[x_{i-1}, x_i]$ the Sup Inf value of are $M_i = 1$ and $m_i = -1$.
Therefore $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = 1(1 - 0) = 1$.
$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = -1(1 - 0) = -1$.
This implies

$$\int_{\underline{0}}^{1} f dx \neq \int_{0}^{\overline{1}} f dx$$

Examples

Example

Show that $f(x) = x^2$ is integrable over $[0, k]$.
Let $P = \{0, \frac{k}{n}, \frac{2k}{n}, \cdots, \frac{nk}{n} = k\}$ be any partition of $[0, 1]$. Let for any sub-interval $[(i-1)\frac{k}{n}, i\frac{k}{n}]$ the Sup Inf value of are $M_i = [(i-1)\frac{k}{n}]^2$ and $m_i = [\frac{ik}{n}]^2$.
Therefore $U(P, x^2) = \sum_{i=1}^{n} M_i \Delta x_i = \frac{k^3}{n^3}\frac{n}{6}(n+1)(2n+1)$.
$L(P, x^2) = \sum_{i=1}^{n} m_i \Delta x_i = \frac{k^3}{n^3}\frac{n}{6}(n-1)(2n-1)$.
This implies

$$\int_{0_-}^{k} x^2 dx = \lim_{n\to\infty} \frac{k^3}{n^3}\frac{n}{6}(n+1)(2n+1) = \frac{k^3}{6}$$

$$\int_{0}^{\bar{k}} f dx = \lim_{n\to\infty} \frac{k^3}{n^3}\frac{n}{6}(n-1)(2n-1) = \frac{k^3}{6}$$

Hence the function x^2 is R-integrable over $[0, k]$.

Related Theorems

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Definition

A partition P^* is said to be a refinement of P if $P^* \supseteq P$, i.e., every point of P is a point of P^*.

Theorem

If P^* is a refiement of a partition P, then for a bounded function f,

1. $L(P^*, f) \geq L(P, f)$, *and*
2. $U(P^*, f) \leq U(P, f)$.

Related Theorems

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Proof.

Suppose first that P^* contains just one point more than P, and let that point is in Δx_i, that is, $x_{i-1} < \xi < x_i$. As the function is bounded over $[a, b]$, it is bounded in every sub-interval Δx_i.
Let w_1, w_2, m_i be the infimum (g.l.b) of f in the intervals $[x_{i-1}, \xi]$, $[\xi, x_i]$, $[x_{i-1}, x_i]$ respectively Clearly
$m_i \leq w_1, m_i \leq w_2$
Therefore, $L(P^*, f) - L(P, f)$
$= w_1(\xi - x_{i-1}) + w_2(x_i - \xi) - m_i(x_i - x_{i-1})$
$= w_1(\xi - x_{i-1}) + w_2(x_i - \xi) - m_i(x_i - \xi + \xi - x_{i-1})$
$= (w_1 - m_i)(\xi - x_{i-1}) + (w_2 - m_i)(x_i - \xi)$
≥ 0
If P^* contains p points more than P, we repeat the above reasoning p times and arrive at $L(P^*, f) \geq L(P, f)$
Similarly, we can prove that $U(P^*, f) \leq U(P, f)$. □

Related Theorems

Theorem

If P^* is a refiement of a partition P contains p points more than P, and function $|f(x)| \leq k$ for all $x \in [a, b]$, then

1. $L(P, f) \leq L(P^*, f) \leq L(P, f) + 2pk\mu$, *and*
2. $U(P, f) \geq U(P^*, f) \geq U(P, f) - 2pk\mu$.

Related Theorems

Proof.

Proceeding as in the above theorem,
$L(P^*, f) - L(P, f) = (w_1 - m_i)(\xi - x_{i-1}) + (w_2 - m_i)(x_i - \xi)$
Since $|f(x)| \leq k$ for all $x \in [a, b]$ therefore
-k $\leq m_i \leq w_1 \leq k$
$\Rightarrow 0 \leq w_1 - m_i \leq 2k$
Similarly $0 \leq w_2 - m_i \leq 2k$
therefore
$L(P^*, f) - L(P, f) \leq 2k(\xi - x_{i-1}) + 2k(x_i - \xi) = 2k\Delta x_i \leq 2k\mu$
Now supposing that each additional point is introduced one by one, by repeating the above reasoning p times we get

$L(P, f) \leq L(P^*, f) \leq L(P, f) + 2pk\mu$.
Similarly the other result may be proved. □

Malik, S. C. and Arora, S., Mathematical Analysis, New Age International Publishers.

https://en.wikipedia.org/wiki/MathematicalAnalysis (Accessed from Internet)