Numericals (Solved)

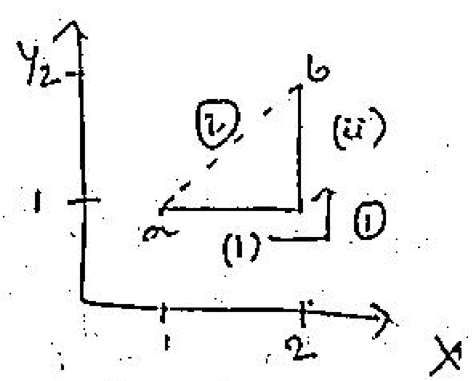
Line integral / Path integral

$$\int_a^b v \cdot dl \qquad \oint v \cdot dl.$$

* 'v' is a vector function
* 'dl' is the infinitesimal displacement vector.
* Integral is to be carried out along a prescribed path from point a to b.
* If the path forms a closed loop, there will be a circle on the integral sign.

Example:

① calculate the line integral of the function $v = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point $a = (1,1,0)$ to point $b = (2,2,0)$ along the paths ① and ② as shown in figure. What is $\oint v \cdot dl$ for the loop that goes from a to b along ① and returns to 'a' along ②?



Solution

For path ① i.e. from $(1,1,0) \xrightarrow{X} (2,1,0) \xrightarrow{Y} (2,2,0)$

We have,

$$dl = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}.$$

Path ① consists of two parts along path
Ⓧ and Ⓨ.

Along X, $dy = dz = 0$.

$$\therefore dl = dx\,\hat{x}, \quad y = 1$$

$$\therefore \oint v \cdot dl = \int \left(y^2\hat{x} + (2x(y+1))\hat{y}\right) \cdot dx\,\hat{x}$$

$$= \int_1^2 y^2\,dx$$

$$= \int_1^2 dx \quad ; \because y = 1.$$

$$= x\Big|_1^2 = 2-1 = 1$$

Along Ⓨ

$$dx = dz = 0$$

$$dl = dy\,\hat{y}, \quad x = 2$$

$$\int v \cdot dl = \int_1^2 2x(y+1)\,dy.$$

$$= 4\int_1^2 (y+1)\,dy$$

$$= 4\left[y^2/2 + y\right]_1^2$$

$$= 4\left[\left(\frac{2^2}{2} - \frac{1}{2}\right) + (2-1)\right]$$

$$= 4\left[\left(2 - \tfrac{1}{2}\right) + 1\right]$$

$$= 4 \times \left(\tfrac{3}{2} + 1\right)$$

$$= \cancel{4}^{2} \times \frac{5}{\cancel{2}}$$

$$= 10$$

$\therefore$ Along path 1.

$$\int_a^b v \cdot dl = 1 + 10 = 11$$

Along path 2

$$x = y \Rightarrow dx = dy \quad \text{\& } dz = 0.$$

$$\therefore dl = dx\,\hat{x} + dx\,\hat{y}.$$

$$v \cdot dl = x^2 \cdot dx + 2x(x+1)\,dx$$

$$= (3x^2 + 2x)\,dx.$$

$$\therefore \oint_a^b v \cdot dl = \int_1^2 (3x^2 + 2x)\,dx = 3\frac{x^3}{3}\Big|_1^2 + 2 \cdot \frac{x^2}{2}\Big|_1^2$$

$$= x^3 + x^2 \Big|_1^2$$

$$= (8-1) + (4-1)$$

$$= 7 + 3$$

$$= 10$$

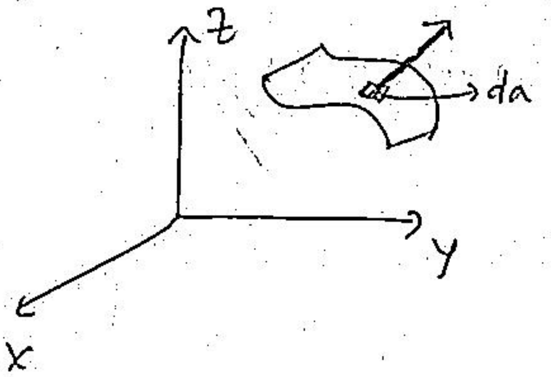
∴ For the loop that goes from a to b along ① and returns to a along 2 is –

$$\oint v \cdot dl = 11 - 10 = 1$$

Surface Integrals

$$\int_S v \cdot da \qquad \oint v \cdot da \text{ or } \iint_S v \cdot da \quad \oiint v \cdot da$$

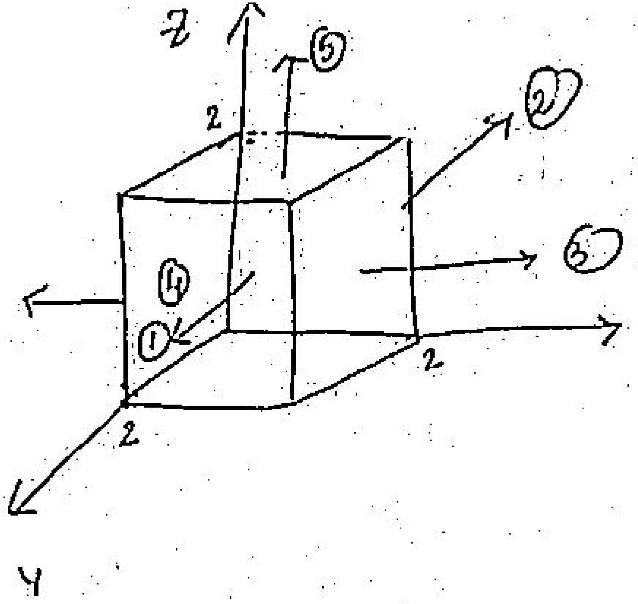
'da' is an infinitesimal patch of area with direction perpendicular to the surface.



Example:

Calculate the surface integral of $v = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$ over five sides (excluding the bottom) of the cubical box as shown in figure. Let 'upward and outward' be the positive direction as indicated by the arrows.

Solution:



For side ①

$x = 2$, $da = dy\,dz\,\hat{x}$; $v \cdot da = 2xz\,dy\,dz$

$= 4z\,dy\,dz$.

$\therefore \int v \cdot da = \int 4z\,dy\,dz = 4\int_0^2 dy \int_0^2 z\,dz$

$= 4\; y\Big|_0^2\; z\Big|_0^2$

$= 4\,[2-0]\,[2-0]$

$= 16$

For side 2

$x = 0$, $da = -dy\,dz\,\hat{x}$, $v \cdot da = -2xz\,dy\,dz = 0$.

$\therefore \int v \cdot da = 0$

For side 3

$y = 2$, $da = dx\,dz\,\hat{y}$, $v \cdot da = (x+2)\,dx\,dz$.

$\int v \cdot da = \int_0^2 (x+2)\,dx \int_0^2 dz = \frac{x^2}{2}\Big|_0^2 + 2x\Big|_0^2 \cdot z\Big|_0^2$

$= (2-0) + (4-0) \times (2-0)$

$= 12$

For side 4

$y = 0, \; da = -dx\,dz\,\hat{y}$

$v \cdot da = -(x+2)\,dx\,dz$

$\int v \cdot da = -\int (x+2)\,dx\,dz$

$= -\int_0^2 (x+2)\,dx \int_0^2 dz$

$= -\left[\int_0^2 x\,dx + \int_0^2 2\,dx * \int_0^2 dz\right]$

$= -\left[\frac{x^2}{2}\Big|_0^2 + 2x\Big|_0^2 * z\Big|_0^2\right]$

$= -\left[\frac{\cancel{4}^2}{\cancel{2}} + 2 \times 2 * 2\right]$

$= -\cancel{8}12$

For side 5

$z = 2, \; da = dx\,dy\,\hat{z}, \; v \cdot da = y(z^2-3)\,dx\,dy$

$\Rightarrow v \cdot da = y\,dx\,dy$.

$\therefore \int v \cdot da = \int y\,dx\,dy$

$= \int_0^2 dx \int_0^2 y\,dy$

$= x\Big|_0^2 \times y^2/2\Big|_0^2 = 2 \times 2 = 4.$

∴ The total surface integral (flux) is

$$\int v \cdot da = 16 + 0 + 12 - 12 + 4 = 20$$

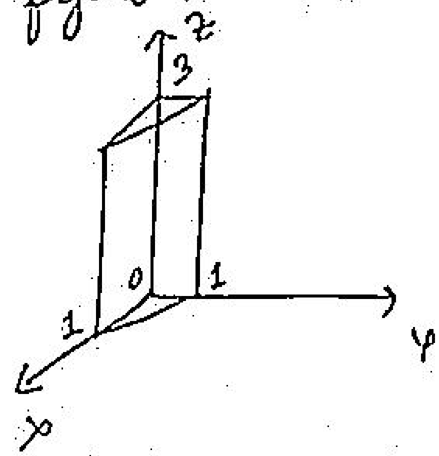
✱ CHECK USING DIVERGENCE THEOREM

Volume Integrals

$$\int_V v\, dV. \qquad dV = dx\,dy\,dz.$$

Example:

Calculate the volume integral of $T = xyz^2$ over the prism as shown in figure.



$y = mx + c.$
At $x = 0, y = 1$
$x = 1, y = 0$
∴ At $x = 0,$
$y = c = 1.$
Again, $m = \frac{dy}{dx} = -1$
∴ $y = -x + 1$
$x = 1 - y$

In x-direction

$y = 1 - \text{slope}$
$= 1 - x$
$x = 1 - y.$

Solution:

$$\iiint T\,dV = \iiint xyz^2\,dx\,dy\,dz$$

$$= \int z^2 dz \int y\,dy \int x\,dx.$$

$$= \int_0^3 z^2 dz \int_0^1 y\,dy \int_0^{1-y} x\,dx$$

$$= \frac{1}{2} \int z^2 dz \int y\,dy \left[(1-y)^2 \right]$$

$$= \frac{1}{2}\int_0^3 x^2 dx \int_0^1 (y^3 - 2y^2 + y)\, dy$$

$$= \frac{1}{2}\int_0^3 x^2 dx \left[\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}\frac{x^3}{3}\Big|_0^3 \left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{2} \times 9 \times \frac{3-8+6}{12}$$

$$= \frac{1}{2} \times \cancel{9}^{3} \times \frac{1}{\cancel{12}_4}$$

$$= \frac{3}{8}$$

4/10

(1) Calculate the line integral of the function $v = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$ from the origin to the point $(1,1,1)$ by three different routes.

(a) $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)$

(b) $(0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1)$

(c) The direct straight line.

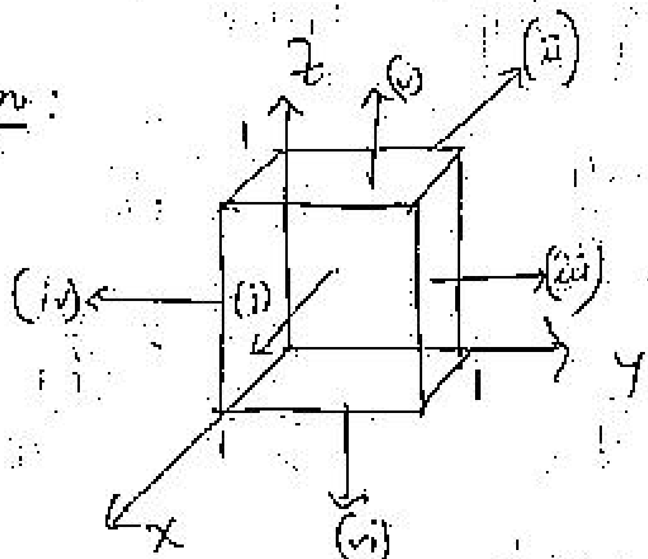
(d) What is the line integral around the closed loop that goes along path a and then b.

Q) Check the divergence theorem using the function.

$$\vec{A} = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$$

Over the unit cube situated at the origin shown in figure.

<u>Solution</u>:



<u>To Prove</u>

Divergence theorem

$$\iiint_V (\vec{\nabla}\cdot\vec{A})\, dV = \oint_S A \cdot dS$$

<u>LHS</u>

$$\iiint_V (\vec{\nabla}\cdot\vec{A})\, dV$$

$$= \iiint_V 2(x+y)\, dV$$

$$= 2\int_0^1\int_0^1\int_0^1 (x+y)\, dx\, dy\, dz$$

$$= 2\left[\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz + \int_0^1\int_0^1\int_0^1 y\,dx\,dy\,dz\right]$$

$$= 2\left[\tfrac{1}{2}\times 1\times 1 + \tfrac{1}{2}\times 1\times 1\right]$$

$$= 2\times\left[\tfrac{1}{2}+\tfrac{1}{2}\right]$$

$$= 2$$

To Calculate RHS, that is surface integral, we need separately consider the six sides of the cube.

(i) $\int \vec{A}\cdot d\vec{s} = \int_0^1\int_0^1 y^2\,dy\,dz = \tfrac{1}{3}$

(ii) $\int \vec{A}\cdot d\vec{s} = -\int_0^1\int_0^1 y^2\,dy\,dz = -\tfrac{1}{3}$.

(iii) $= \int \vec{A}\cdot d\vec{s} = +\int_0^1\int_0^1 (2x+z^2)\,dx\,dz = \tfrac{4}{3}$.

(iv) $\int \vec{A}\cdot d\vec{s} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\tfrac{1}{3}$.

(v) $\int \vec{A}\cdot d\vec{s} = +\int_0^1\int_0^1 2y\,dx\,dy = 1$

(vi) $\int \vec{A}\cdot ds = -\int_0^1\int_0^1 0\,dx\,dy = 0$

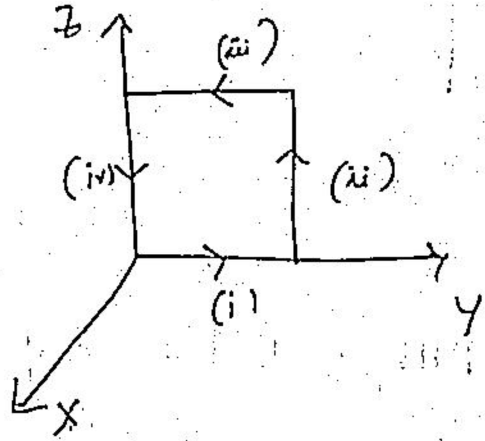
$\therefore$ Total surface integral $= \oint \vec{A}\cdot d\vec{s} = \cancel{2}$

$\tfrac{1}{3} - \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{1}{3} + 1 = 2.$

Thus Divergence theorem is proved.

Q. Suppose $\vec{A} = (2xz+3y^2)\hat{j} + (4yz^2)\hat{k}$.
Check Stoke's theorem for the square surface shown in figure.

<u>Solution.</u>



Here,

$\vec{\nabla}\times\vec{A}$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left((2xz+3y^2)\hat{j} + (4yz^2)\hat{k}\right)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz+3y^2 & 4yz^2 \end{vmatrix}$$

$$= \hat{i}\,(4z^2 - 2x) + \hat{k}\,4yz^2,$$

Again $x = 0$ for this surface,

$$\therefore \int (\nabla \times A)\cdot ds = \int_0^1 \int_0^1 4z^2\, dy\, dz = \frac{4}{3}.$$

To Calculate line Integral, we compute separately

(i) $x=0,\ z=0,\ \vec{A}\cdot\vec{dl} = 3y^2 dy \Rightarrow \int \vec{A}\cdot\vec{dl} = \int_0^1 3y^2 dy = 1$

(ii) $x=0,\ y=1,\ \vec{A}\cdot\vec{dl} = 4z^2 dy \Rightarrow \int \vec{A}\cdot\vec{dl} = \int_0^1 4z^2 dz = \frac{4}{3}$

(iii) $x=0,\ z=1,\ \vec{A}\cdot\vec{dl} = 3y^2 dy \Rightarrow \int \vec{A}\cdot\vec{dl} = \int_0^1 3y^2 dy = -1$

(iv) $x=0,\ y=0,\ \vec{A}\cdot\vec{dl} = 0 \Rightarrow \int \vec{A}\cdot\vec{dl} = \int_0^1 0\, dz = 0$

$\therefore \oint \vec{A}\cdot\vec{dl} = 1 + 4/3 - 1 + 0 = 4/3$.

Thus $\int (\nabla \times A)\cdot ds = \oint \vec{A}\cdot\vec{dl}$