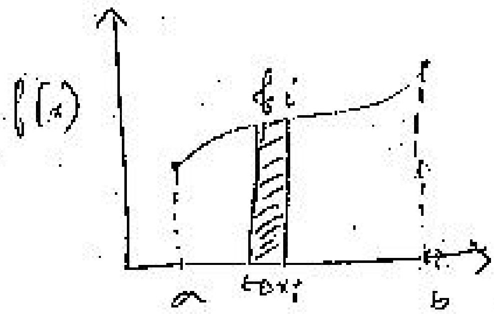
Integral and Differential form of Faraday's law

1) Line Integral
2) Surface Integral
3) Volume Integral

Line Integral : It is the integral of some function along a curve.

Let us consider a function $f(x)$ which is continuous over x, between the limits a and b.

To find the line integral from a to b for the function $f(x)$.



We divide the interval from a to b into 'n' number of times small segments of length $\Delta x \to 0$

$$\boxed{\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f_i \, \Delta x_i}$$

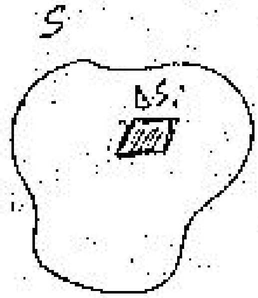
If $\oint F.dx = 0 \longrightarrow F$ is conservative

Eg (i) Work done on a charged particle travelling along a curve

(ii) $\oint E.dl = V$ (line integral of electric field intensity over a closed path)

(b) <u>Surface Integral</u>: It is the integration over surfaces.

To define the surface integral of $\vec{F}$ we divide the surface into small surfaces (surface elements) ΔS.



$$\boxed{\iint \vec{F} \cdot ds = \lim_{\substack{n \to \infty \\ \Delta S_i \to 0}} \sum_{i=1}^{n} \vec{F_i}\, \Delta \vec{S_i}}$$

[$\Delta S_i \to$ surface element]

$F_i \to$ Field quantity for ΔS_i.

(c) <u>Volume Integral</u>: It is the integration over a volume.

To define volume integral, divide a given volume into n number of small volume elements Δv.

$$\boxed{\iiint \vec{F} \cdot d\hat{v} = \lim_{\substack{n \to \infty \\ \Delta v_i \to 0}} \sum_{i=1}^{n} \vec{F_i} \cdot \Delta v_i}$$

[$\Delta v_i \to$ volume integral]

<u>Applications</u>:

Used in the basic laws of electromagnetic fields.

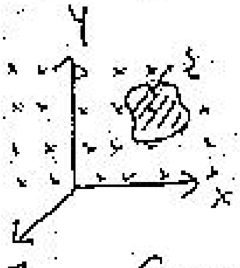
Ex:- Divergence theorem ✓
Stoke's theorem ✓
Green's theorem ✓
Ampere's law
Coulomb's law

Imp : $\vec{\nabla}\times\vec{F}$, $\vec{\nabla}\cdot\vec{F}$

Surface integral over a closed surface S.

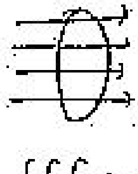
Divergence theorem / Gauss's theorem

$$\iint_S \vec{F}\cdot\vec{n}\, ds = \iiint \nabla\cdot\vec{F}\, dV$$



$$= \iiint \vec{\nabla}\cdot\vec{F}\, dx\,dy\,dz$$

Gauss's theorem:



$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\, ds}_{\text{Flux of an E field.}} = \underbrace{\iiint \vec{\nabla}\cdot\vec{F}\, dx\,dy\,dz}_{\text{Amount of charge contained within the volume.}}$$

Line integral around a closed curve

Green's theorem

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \iint_D \nabla\cdot\vec{F}\, dA$$

$$\int \longrightarrow \iint$$

$$\boxed{\oint \vec{F}\cdot d\vec{r} = \oint (M\,dx + N\,dy) = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

Stoke's theorem $\int \to \iint$

$$\boxed{\oint \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds.}$$

Ex: ① Find the value of the integral $\oiint \vec{r}\cdot\vec{n}\, ds$ over a closed surface S bounding a volume V. Ans.

② Find the value of the line integral $\int \vec{F}(r)\cdot d\vec{r}$ for $\vec{F}(\vec{r}) = x^2\hat{i} + y^2\hat{j}$ along C which is a straight line joining (0,0) to (1,1). (Green's theorem ??

$$\int_C x^2 dx + y^2 dy = ? \longrightarrow ①$$

Eq^n of straight line, $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = t$

$\int (2t^2) dt = t^3/3 \Big|_0^t$ $\Rightarrow x - 0 = y - 0$

8) The surface integral $\iint \vec{F}\cdot\hat{n}\,ds$ over the surface S of the sphere $x^2+y^2+z^2=9$ where $\vec{F}=(x+y)\hat{i}+(x+z)\hat{j}+(y+z)\hat{k}$ and $\hat{n}$ is the unit outward surface normal, is $\underline{\underline{\quad ?\quad}}$?

[Ans : 226.19]

Which theorem ?

5) A scalar potential ϕ has the gradient $\nabla\phi = yz\hat{i}+xz\hat{j}+xy\hat{k}$. Find the integral $\int_c \nabla\phi\cdot d\vec{r}$ on the curve $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

The curve c is parameterized as —

[Ans: 726]

$$x=t,\quad y=t^2,\quad z=3t^2,\quad \underline{1\le t\le 3}$$

Sol:

$$\int \vec{\nabla}\phi\cdot d\vec{r}$$

$$=\int\left(yz\hat{i}+xz\hat{j}+xy\hat{k}\right)\cdot\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)$$

$$=\int yz\,dx + xz\,dy + xy\,dz \quad \text{———} (1)$$

$$x=t \;;\; y=t^2 \;;\; z=3t^2$$

$$\Rightarrow dx=dt \quad \Rightarrow dy=2t\,dt \quad dz=6t\,dt$$

$$(1)=\int_1^3\left(3t^4\,dt + 3t^3\cdot 2t\,dt + t^3\cdot 6t\,dt\right)$$

$$=\int_1^3\left(3t^4\,dt + 6t^4\,dt + 6t^4\,dt\right)$$

$$=\int_1^3 15t^4\,dt = 15\left.\frac{t^5}{5}\right|_1^3 = \left.3t^5\right|_1^3$$

$$=3\left(3^5-1\right)=726.$$

Faraday's law:

2nd law: $e = -\frac{d\phi}{dt}$ ⟶ ① ; $\phi \Rightarrow$ Magnetic flux

'−' → direction of current

Magnetic flux is the scalar product of magnetic field $\vec{B}$ and vector area dA.

$$\phi = \int_s \vec{B} \cdot d\vec{A} \longrightarrow ②$$

From definition of e.m.f (electromotive force)

$$e = \oint_c \vec{E} \cdot d\vec{s} \longrightarrow ③$$

[Here ds → line integral]

[E → electric field due to induced emf]

Putting eqn ② & ③ in eqn ①.

$$\boxed{e = -\frac{d}{dt}\int_s \vec{B} \cdot d\vec{A} = \oint_c \vec{E} \cdot d\vec{s}} \longrightarrow ④$$

Integral form of Faraday's law of em. induction

Again, $$\oint_c \vec{E} \cdot d\vec{s} = -\int_s \frac{d\vec{B}}{dt} \cdot d\vec{A} \longrightarrow ⑤$$

Using Stoke's theorem (Curl theorem)

$$\oint_c \vec{E} \cdot d\vec{s} = \int_s (\nabla \times \vec{E}) \cdot d\vec{A} \longrightarrow ⑥$$

from eqn ⑤ and ⑥

$$-\int_s \frac{dB}{dt} \cdot d\vec{A} = \int_s (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} \longrightarrow ⑦$$

Comparing both sides of equation (7):

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{dB}{dt}} \quad \text{——— (8)}$$

$$\text{Curl } \vec{E} = -\frac{d\vec{B}}{dt}$$

This is the differential form of Faraday's Law