2

EQUATIONS OF FIRST ORDER AND FIRST DEGREE

2·1. Introduction. There are two standard forms of differential equations of first order and first degree, namely,

(*i*) $dy/dx = f(x, y)$ (*ii*) $M(x, y)\, dx + N(x, y)\, dy = 0$.

In what follows we shall see that an equation in one of these forms may readily be written in the other form. It will be assumed that the necessary conditions for the existence of solutions are satisfied. We now discuss various methods to solve such equations.

2·2. Separation of variables. If a differential equation of the first order and first degree is of the form

$$f_1(x)\, dx = f_2(y)\, dy, \qquad \text{...(1)}$$

where $f_1(x)$ is a function of x only and $f_2(y)$ is a function of y only, then we say that the variables are separable in the given differential equation.

Such equations are solved by integrating both sides of (1) and adding an arbitrary constant of integration to any one of the two sides. Thus, solution of (1) is

$$\int f_1(x)\, dx = \int f_2(y)\, dy + C, \qquad \text{...(2)}$$

Remark. To simplify the solution (2), the constant of integration C can be selected in any suitable form. For example, C can be replaced by $C/3$, $\log C$, $\tan C$, $\tan^{-1} C$, e^C etc.

Examples of Type 1 based on Art. 2.2

Ex. 1. *Solve* $dy/dx = e^{x-y} + x^2 e^{-y}$. **(Vikram 93 ; Rajpur 91 ; Jodhpur 92 ; Lucknow 98 ; Meerut 88 ; Punjab 94 ; Agra 95)**

Sol. For separating variables, we re-write given equation as

$$dy/dx = e^{-y}(e^x + x^2) \text{ or } e^y\, dy = (x^2 + e^x)\, dx.$$

Integrating, $e^y = (1/3)\, x^3 + e^x + c$.

Ex. 2. *Solve* $\log\,(dy/dx) = ax + by$.

(Berhampur 96 ; Allahabad 93 ; Bihar 94 ; Magadh 98)

Sol. Re-writing the given equation, we get

$$dy/dx = e^{ax+by} = e^{ax}e^{by} \text{ or } e^{-by}\,dy = e^{ax}\,dx.$$

Integrating, $\quad -(1/b)e^{-by} = (1/a)e^{ax} + c.$

Ex. 3. *Solve* $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$. **(Meerut 93 ; Bilaspur 88 ; Delhi B.Sc.(G) 94 ; Rajasthan 95 ; Agra 93 ; Indore 93)**

Sol. The given equation can be re-written as

$$(a+x)\frac{dy}{dx} = y - ay \text{ or } \frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

or
$$\frac{dx}{x+a} = \left[\frac{a}{1-ay} + \frac{1}{y}\right] dy.$$

Integrating, $\log(x+a) = -\log(1-ay) + \log y + \log c,$

or
$$\log(x+a) = \log\left[\frac{cy}{1-ay}\right] \quad \text{or} \quad x + a = \frac{cy}{1-ay}$$

or $\quad (x+a)(1-ay) = cy$, which is the required solution.

Ex. 4. *Solve* $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0.$ **(Delhi B.Sc, (G) 86; Kanpur 97 ; Andhra 91 ; Vikram 91 ; Banaras 91 ; Indore 89)**

Sol. Separating the variables, we get

$$\frac{3e^x}{1-e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0.$$

Integrating, $-3\log(1-e^x) + \log(\tan y) = \log c.$

or $\quad \log(\tan y) = \log(1-e^x)^3 + \log c, \;$ or $\; \tan y = c(1-e^x)^3.$

Ex. 5. *Solve* $\sqrt{(1+x^2+y^2+x^2y^2)} + xy\,(dy/dx) = 0.$

Sol. Re-writing the given differential equation, we have

$$\sqrt{[(1+x^2)(1+y^2)]} + xy\,(dy/dx) = 0 \text{ or } \sqrt{(1+x^2)}\sqrt{(1+y^2)} + xy\,(dy/dx) = 0$$

or
$$\frac{\sqrt{(1+x^2)}\,dx}{x} + \frac{y\,dy}{\sqrt{(1+y^2)}} = 0 \quad \text{or} \quad \frac{(1+x^2)\,dx}{x\sqrt{(1+x^2)}} + \frac{y\,dy}{\sqrt{(1+y^2)}} = 0.$$

Integrating,
$$\int \frac{dx}{x\sqrt{(1+x^2)}} + \int \frac{x\,dx}{\sqrt{(1+x^2)}} + \int \frac{y\,dy}{\sqrt{(1+y^2)}} = C. \qquad \ldots(1)$$

Now,
$$\int \frac{dx}{x\sqrt{(1+x^2)}} = \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{1+(1/t)^2}}, \text{ putting } x = \frac{1}{t}$$

$$= -\int \frac{dt}{\sqrt{(t^2+1)}} = -\log\{t + \sqrt{(t^2+1)}\}$$

$$= -\log\left\{\frac{1}{x} + \sqrt{\left(\frac{1}{x^2}+1\right)}\right\} = -\log\left\{\frac{1+\sqrt{(1+x^2)}}{x}\right\}$$

$$= \log x - \log \{1 + \sqrt{(1+x^2)}\}. \quad ...(2)$$

Again, $\int \frac{x\,dx}{\sqrt{(1+x^2)}} = \int \frac{t\,dt}{2\sqrt{t}}$, putting $1 + x^2 = t$

$$= \frac{1}{2} \int t^{-1/2}\,dt = t^{1/2} = (1+x^2)^{1/2}. \quad ...(3)$$

Similarly, $\int \frac{y\,dy}{\sqrt{(1+y^2)}} = (1+y^2)^{1/2}$. ...(4)

Using (2), (3) and (4), (1) gives the required solution as

$$\log x - \log \{1 + \sqrt{(1+x^2)}\} + (1+x^2)^{1/2} + (1+y^2)^{1/2} = C.$$

Ex. 6. *Solve* $dy/dx = e^{x+y} + x^2 e^{x^3+y}$.

Sol. From given equation, $dy/dx = e^y (e^x + x^2 e^{x^3})$.

or $$e^{-y}\,dy = (e^x + x^2 e^{x^3})\,dx. \quad ...(1)$$

Integrating (1), $\int e^{-y}\,dy = \int e^x\,dx + \int x^2 e^{x^3}\,dx$

or $$-e^{-y} = e^x + (1/3) \int e^t\,dt + c, \text{ putting } x^3 = t$$

or $$-e^{-y} = e^x + (1/3)\,e^t + c = e^x + (1/3)\,e^{x^3} + c.$$

Ex. 7. *If* $dy/dx = e^{x+y}$ *and it is given that for* $x = 1, y = 1$ *; find* y *when* $x = -1$.

Sol. Given equation is $e^{-y}\,dy = e^x\,dx$.

Integrating it, $-e^{-y} = e^x + c$. ...(1)

Putting $x = 1, y = 1$ in (1), $-e^{-1} = e + c$ so that $c = -e^{-1} - e$.

Hence (1) becomes $-e^{-y} = e^x - e^{-1} - e$. ...(2)

Putting $x = -1$ in (2), we obtain

$$-e^{-y} = e^{-1} - e^{-1} - e \text{ so that } y = -1. \quad \textbf{Ans.}$$

Exercise 2(A)

1. $(e^x + 1)\,y\,dy = (y+1)\,e^x\,dx$. **(Agra 96)** **Ans.** $(e^x + 1)(y+1) = ce^y$
2. $(dy/dx) - y \tan x = -y \sec^2 x$. **Ans.** $y \cos x = ce^{-\tan x}$
3. $x\sqrt{(1+y^2)}\,dx + y\sqrt{(1+x^2)}\,dy = 0$. **(Bangalore 96)** **Ans.** $\sqrt{(1+x^2)} + \sqrt{(1+y^2)} = C$
4. $(2ax + x^2)(dy/dx) = a^2 + 2ax$. **(Kanpur 96)** **Ans.** $x(x+2a)^3 = C\,e^{(2y/a)}$
5. $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$. **(Kanpur 93 ; Agra 96 ; Meerut 83 ; Vikram 75)** **Ans.** $x^2 \log x + c = y \sin y$
6. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$. **(Lucknow 92)** **Ans.** $(\sin x)(e^y + 1) = c$
7. (*a*) $\sqrt{(a+x)}\,(dy/dx) + x = 0$. **(Rohilkhand 95 ; Agra 91 ; Bundelkhand 98)** **Ans.** $y + (2/3)(x - 2a)(a+x)^{1/2} = c$

   (*b*) $dy/dx = \sqrt{(1-y^2)}/(1-x^2)$ **(Dibrugarh 96 ; Bangalore 96)** **Ans.** $\sin^{-1} x = \sin^{-1} y + c$

8. $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0.$ **Ans.** $\log (x/y) - (x+y)/(xy) = c$

9. $(xy^2 + x) dx + (yx^2 + y)\, dy = 0.$ **Ans.** $(x^2+1)(y^2+1) = c$

10. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0.$ **Ans.** $\tan x \tan y = c$

11. $(1+x) y\, dx + (1+y)\, x\, dy = 0.$ **Ans.** $x + y + \log (xy) = c$

12. Find the function 'f' which satisfies the equation $df/dx = 2f$, given that $f(0) = e^3$. **Ans.** $f = e^{2x+3}$

13. $(1 - x^2)(1 - y)\, dx = xy\, (1 + y) dy.$

(Agra 92 ; Meerut 95 ; Jabalpur 93 ; Guwahati 96 ; Vikram 92)

Ans. $\log [x(1-y)^2] = \frac{1}{2}(x^2 - y^2) - 2y + c$

14. $x^2 (y+1)\, dx + y^2 (x-1)\, dy = 0.$ **(Agra 94 ; Meerut 85)**

Ans. $x^2 + y^2 + 2(x - y) + 2 \log \{(x-1)(y+1)\} = c$

15. $(dy/dx) \tan y = \sin (x+y) + \sin (x-y).$ **Ans.** $2 \cos x + \sec y = c$

16. $\operatorname{cosec} x \log y\, dy + x^2 y^2\, dx = 0.$

Ans. $(2 - x^2) \cos x + 2x \sin x - (1/y)(1 + \log y) = c$

17. $y - x\dfrac{dy}{dx} = 3\left(1 + x^2 \dfrac{dy}{dx}\right).$ **Ans.** $(y-3)(1+3x) = cx$

18. $\cos y \log (\sec x + \tan x)\, dx = \cos x \log (\sec y + \tan y)\, dy.$ **(Kanpur 94)**

Ans. $\log \dfrac{\sec x + \tan x}{\sec y + \tan y} \log \{(\sec x + \tan x)(\sec y + \tan y)\} = c$

2·3. Transformation of some equations in the form in which variables are separable. Equations of the form **[Nagpur 2003]**

$$dy/dx = f(ax + by + c) \quad \text{or} \quad dy/dx = f(ax + by)$$

can be reduced to an equation in which variables can be separated. For this purpose we use the substitution $ax + by + c = v$ or $ax + by = v$.

Examples of Type 2 based on Art. 2.3

Ex. 1. *Solve* $dy/dx = (4x + y + 1)^2$. **(Jiwaji 93 ; Vikram 90 ; Agra 91 ; Raj. 92 ; Lucknow 93 ; Ravishankar 93)**

Sol. Let $\qquad 4x + y + 1 = v.$ ...(1)

Differentiating (1) with respect to x, we get

$$4 + \frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dv}{dx} - 4. \qquad \text{...(2)}$$

Using (1) and (2), the given equation becomes

$$\frac{dv}{dx} - 4 = v^2 \quad \text{or} \quad \frac{dv}{dx} = 4 + v^2.$$

Now separating variables x and v, $dx = \dfrac{dv}{4 + v^2}$.

Integrating, $x + c = (1/2) \tan^{-1} (v/2)$, where c is an arbitrary constant.

or $\quad 2x+2c=\tan^{-1}(v/2)$ or $v=2\tan(2x+2c)$

or $\quad 4x+y+1=2\tan(2x+2c)$, using (1)

Ex. 2. *Solve* $(x+y)^2(dy/dx)=a^2$. **(Meerut 97 ; Indore 98 ; I.A.S. (Preliminary) 94 ; Delhi B.Sc. (G) 97 ; Ravishankar 92)**

Sol. Let $\quad x+y=v$. ...(1)

Differentiating, $1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$ or $\dfrac{dy}{dx}=\dfrac{dv}{dx}-1$. ...(2)

Using (1) and (2), the given equation becomes

$$v^2\left(\frac{dv}{dx}-1\right)=a^2 \quad \text{or} \quad v^2\frac{dv}{dx}=a^2+v^2$$

or $\quad dx=\dfrac{v^2}{v^2+a^2}dv$ or $dx=\left[1-\dfrac{a^2}{a^2+v^2}\right]dv$.

Inegrating, $x+c=v-a^2\times\dfrac{1}{a}\tan^{-1}\dfrac{v}{a}$ where c is arbitrary constant

or $\quad x+c=x+y-a\tan^{-1}\left(\dfrac{x+y}{a}\right)$ or $y-a\tan^{-1}\left(\dfrac{x+y}{a}\right)=c$.

Ex. 3. *Solve* $dy/dx=\sec(x+y)$ **(Meerut 89, 91)**

or $\quad \cos(x+y)\,dy=dx$. **(Meerut 89 ; Kanpur 92)**

Sol. Let $\quad x+y=v$,

so that $\quad dy/dx=(dv/dx)-1$. ...(1)

Using (1), the given equation becomes

$$\frac{dv}{dx}-1=\sec v \quad \text{or} \quad \frac{dv}{dx}=1+\frac{1}{\cos v}$$

or $\quad dx=\dfrac{\cos v}{1+\cos v}dv=\dfrac{2\cos^2\frac{1}{2}v-1}{1+2\cos^2\frac{1}{2}v-1}dv$ or $dx=(1-\frac{1}{2}\sec^2\frac{1}{2}v)\,dv$.

Integrating, $x+c=v-\tan\frac{1}{2}v$ or $y-\tan\frac{1}{2}(x+y)=c$.

Ex. 4. *Solve* $dy/dx=\sin(x+y)+\cos(x+y)$. **(Garhwal 94)**

Sol. Let $\quad x+y=v$. ...(1)

Differentiating, $\quad 1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$ or $\dfrac{dy}{dx}=\dfrac{dv}{dx}-1$. ...(2)

Using (1) and (2), the given equation becomes

$$\frac{dv}{dx}-1=\sin v+\cos v \quad \text{or} \quad \frac{dv}{dx}=1+\sin v+\cos v. \qquad ...(3)$$

But $1+\sin v+\cos v=1+2\sin(v/2)\cos(v/2)+2\cos^2(v/2)-1$

$$=2\cos^2(v/2)\,[1+\tan v/2].$$

$\therefore$ by (3), $dx = \dfrac{dv}{2\cos^2(v/2)\,[1+\tan(v/2)]} = \dfrac{\frac{1}{2}\sec^2(v/2)\,dv}{1+\tan(v/2)}$.

Integrating, $x + c = \log[1+\tan(v/2)]$

or $x + c = \log[1 + \tan\{(x+y)/2\}]$, on using (1).

Ex. 5. *Solve* $(x+y)(dx - dy) = dx + dy$. **(Calcutta 95 ; Agra 87 ; Vikram 92 ; Kumaun 90 ; Ravishankar 92)**

Sol. Re-writing the given equation, we get

$$(x+y-1)\,dx = (x+y+1)\,dy \quad \text{or} \quad \frac{dy}{dx} = \frac{x+y-1}{x+y+1}. \qquad \text{...(1)}$$

Let $x + y = v$. ...(2)

So that as usual $dy/dx = (dv/dx) - 1$. ...(3)

Using (2) and (3), (1) becomes

$$\frac{dv}{dx} - 1 = \frac{v-1}{v+1} \quad \text{or} \quad \frac{dv}{dx} = \frac{2v}{v+1} \quad \text{or} \quad 2\,dx = (1 + 1/v)\,dv.$$

$\therefore$ Integrating, $2x + c = v + \log v$ or $x - y + c = \log(x+y)$.

Ex. 6. *Solve* $\dfrac{dy}{dx} = \dfrac{4x+6y+5}{3y+2x+4}$. **(Raj. 95 ; Meerut 90 ; Calcutta 95 ; Delhi B.Sc. (H) 97, 2002; Karnataka 95)**

Sol. The given equation may be re-written as

$$\frac{dy}{dx} = \frac{2(2x+3y)+5}{(2x+3y)+4}. \qquad \text{...(1)}$$

$\therefore$ We take $2x + 3y = v$. ...(2)

Differentiating, $2 + 3\dfrac{dy}{dx} = \dfrac{dv}{dx}$ or $\dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{dv}{dx} - 2\right)$. ...(3)

Using (2) and (3), (1) gives

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{2v+5}{v+4} \quad \text{or} \quad \frac{dv}{dx} = \frac{3(2v+5)}{v+4} + 2 = \frac{8v+23}{v+4}$$

or $$\frac{dx}{dv} = \frac{v+4}{8v+23} = \frac{\frac{1}{8}(8v+23) + \left(4 - \frac{23}{8}\right)}{8v+23} = \left[\frac{1}{8} + \frac{9}{8(8v+23)}\right]$$

Separating variables, $dx = \left[\dfrac{1}{8} + \dfrac{9}{8(8v+23)}\right]dv$.

Integrating, $x + c = (v/8) + (9/64)\log(8v+23)$

or $8x + 8c = 2x + 3y + (9/8)\log(16x + 24y + 23)$

[using (2) and multiplying by 8]

or $3y - 6x + (9/8)\log(16x + 24y + 23) = 8c$

or $y - 2x + (3/8)\log(16x + 24y + 23) = c'$,

where c' $(= 8c/3)$ is an arbitrary constant.

Exercise 2(B)

1. $dy/dx = (x+y)^2$. (Nagpur 2002) **Ans.** $x + c = \tan^{-1}(x+y)$
2. $dy/dx + 1 = e^{x+y}$. (Calcutta 96) **Ans.** $x + e^{-(x+y)} = c$
3. $(2x+y+1)\,dx + (4x+2y-1)\,dy = 0$. **Ans.** $2y + x + \log(2x+y-1) = c$
4. $(x-y-2)\,dx - (2x-2y-3)\,dy = 0$. **Ans.** $x - 2y - \log(x-y-1) = c$
5. $(x+y+1)(dy/dx) = 1$. **(Meerut 95 ; Jiwaji 81 ; Rohilkhand 81 ; Delhi B.Sc. (G) 91 ; Dibrugarh 95 ; Rohilkhand 97)** **Ans.** $x + y + 2 = ce^y$
6. $\sin^{-1}(dy/dx) = x + y$. **(Meerut 96)** **Ans.** $-2/(x+c) = 1 + \tan\frac{1}{2}(x+y)$
7. $(2x+4y+3)(dy/dx) = 2y + x + 1$. **Ans.** $4x + 8y + 5 = ce^{4(x-2y)}$
8. $\dfrac{4x+6y+5}{3y+2x+4} \cdot \dfrac{dy}{dx} = 1$. **Ans.** $(2/7)(2x+3y) - (9/49)\log(14x+21y+22) = x + c$
9. $dv/dx = (x-y+3)/(2x-2y+5)$. **Ans.** $x - 2y + \log(x-y+2) = c$
10. $(2x+2y+3)\,dy - (x+y+1)\,dx = 0$ or $dy/dx = (x+y+1)/(2x+2y+3)$. **(Lucknow 98 ; Agra 95 ; Bihar 88 ; Kanpur 86 ; Meerut 94)** **Ans.** $x + y + (4/3) = ce^{3(x-2y)}$
11. $(x-y)^2(dy/dx) = a^2$. **(Delhi 99)** **Ans.** $y + c = (a/2)\log\{(x-y-a)/(x-y+a)\}$
12. $\dfrac{x+y-a}{x+y-b} \cdot \dfrac{dy}{dx} = \dfrac{x+y+a}{x+y+b}$. **Ans.** $(b-a)^2 \log\{(x+y)^2 - ab\} = 2(x-y) + c$
13. $dy/dx = \cos(x+y)$. **Ans.** $x + c = \tan\{(x+y)/2\}$
14. If $dy/dx = e^{x+y}$ and it is given that for $x = 1$, $y = 1$, find y when $x = -1$. **Ans.** -1
15. $dy/dx = (x+y+1)/(x+y-1)$ when $y = (1/3)$ at $x = (2/3)$. **Ans.** $\log(x+y) = y - x - (1/3)$
16. $(x+y-1)\,dy = (x+y)\,dx$. **Ans.** $2(y-x) - \log(2x+2y-1) = c$
17. $dy/dx = (x-y+3)/(2x-2y+5)$. **Ans.** $x - y + 2 = ce^{2y-x}$

2·4. Homogeneous equations. Definition. A differential equation of first order and first degree is said to be homogeneous if it can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

2·5. Working rule for solving homogeneous equations.

Let the given equation be homogeneous. Then, by definition, the given equation can be put in the form

$$dy/dx = f(y/x). \qquad \ldots(1)$$

To solve (1), let $y/x = v$, *i.e.*, $y = vx$. ...(2)

Differentiating with respect to x, (2) gives

$$dy/dx = v + x\,(dv/dx). \qquad \ldots(3)$$

Integrating, $\log x = -v^{-1/2} - \log v + \log c$ or $\log\left(\frac{xv}{c}\right) = -\frac{1}{\sqrt{v}}$.

or $\quad \log\left(\frac{xy}{cx}\right) = \frac{-1}{\sqrt{(y/x)}} \quad$ or $\quad \log(y/c) = -\sqrt{(x/y)}$

or $\quad y/c = e^{-\sqrt{(x/y)}}$ so that $y = ce^{-\sqrt{(x/y)}}$

Exercise 2(C)

1. $(x^2 + y^2)\, dx - 2xy\, dy = 0$. **(Delhi Hons. 92; Vikram 62 ; Utkal 67)** **Ans.** $x^2 - y^2 = cx$
2. $y^2 + x^2 \dfrac{dy}{dx} = xy \dfrac{dy}{dx}$. **Ans.** $y = ce^{y/x}$
3. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$. **Ans.** $(x - y)^2 = cxe^{-y/x}$
4. $dy/dx = y/x + \sin(y/x)$. **Ans.** $\tan(y/2x) = cx$
5. $(x^2 + y^2)(dy/dx) = xy$. **(Kerala 2001)** **Ans.** $y = c\, e^{x^2/2y^2}$
6. $(x^2 - y^2)\, dy = 2xy dx$. **Ans.** $y = c(x^2 + y^2)$
7. $(x^2 - y^2)\, dx + 2\, xy\, dy = 0$. **Ans.** $x^2 + y^2 = cx$
8. $y^2\, dx + (xy + x^2)\, dy = 0$. **Ans.** $2y + x = cxy^2$
9. $x\,(dy/dx) + (y^2/x) = y$. **(Delhi 96, 97 ; Dibrugarh 96)** **Ans.** $x = ce^{x/y}$
10. $x^2 y\, dx - (x^3 + y^3)\, dy = 0$. **(Andhra 2003 ; Bangalore 95)** **Ans.** $y^3 = ce^{x^3/y^3}$
11. $(x + y)\, dy + (x - y)\, dx = 0$ or $y - x\,(dy/dx) = x + y\,(dy/dx)$
    or $\quad y - xp = x + yp,\ p = dy/dx$. **(Delhi Hons. 94 ; Rajpur 95)**
    Ans. $\tan^{-1}(y/x) + (1/2)\log(x^2 + y^2) = c$
12. $x\,(x - y)\, dy + y^2\, dx = 0$. **Ans.** $y = ce^{y/x}$
13. $x\,(x - y)\, dy = y\,(x + y)\, dx$. **(Dibrugarh 95)** **Ans.** $xy = ce^{-x/y}$
14. $x \sin(y/x)(dy/dx) = y \sin(y/x) - x$. **(Nagpur 2002)** **Ans.** $x = ce^{\cos(y/x)}$
15. $x^2\, dy + y\,(x + y)\, dx = 0$. **Ans.** $y + 2x = cx^2 y$
16. $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2 y)\, dy$. **Ans.** $x^2 - y^2 = c\,(x^2 + y^2)^2$
17. $2\,(dy/dx) = [y\,(x + y)/x^2]$ or $2\,(dy/dx) - (y/x) = y^2/x^2$. **Ans.** $(y - x)^2 = cxy^2$
18. $(x^3 - 2y^3)\, dx + 3xy^2 dy = 0$. **Ans.** $x^3 + y^3 = cx^2$
19. $dy/dx = (xy^2 - x^2 y)/x^3$. **Ans.** $x^2 y = c\,(y - 2x)$

2·6. Equations reducible to homogeneous form.

Equations of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}, \text{ where } \frac{a}{a'} \neq \frac{b}{b'}, \qquad ...(1)$$

can be reduced to homogeneous form as explained below.

Take $\qquad x = X + h$ and $y = Y + k, \qquad ...(2)$

where X and Y are new variables ... be so chosen that the resulting equation in terms of X and Y may become homogeneous.

From (2), $dx = dX$ and $dy = dY$ $\therefore$ $dy/dx = dY/dX$. ...(3)

Using (2) and (3), (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}. \quad ...(4)$$

In order to make (4) homogeneous, choose h and k so as to satisfy the following two equations

$$ah+bk+c=0 \text{ and } a'h+b'k+c'=0. \quad ...(5)$$

Solving (5), $h = \dfrac{bc'-b'c}{ab'-a'b}$ and $k = \dfrac{ca'-c'a}{ab'-a'b}$. ...(6)

Given that $a/a' \neq b/b'$ $\therefore$ $(ab'-a'b) \neq 0$. Hence h and k given by (6) are meaningful, *i.e.* h and k will exist.

Now h and k are known. So from (2), we get

$$X = x-h \text{ and } Y = y-k. \quad ...(7)$$

In view of (5), (4) reduces to

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} = \frac{a+b(Y/X)}{a'+b'(Y/X)},$$

which is surely homogeneous equation in X and Y and can be solved by putting $Y/X = v$ as usual. After getting solution in terms of X and Y, we remove X and Y by using (7) and obtain solution in terms of the original variables x and y.

Examples of Type 4 based on Art. 2.6

Ex. 1. *Solve* $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$. **(Agra 96 ; Bangalore 96 ; Kiwali 92 ; Delhi B.Sc. (G) 93 ; Gorakhpur 91 ; Lucknow 93)**

Sol. Take $x = X+h, y = Y+k,$ so that $dy/dx = dY/dX$. ...(1)

$\therefore$ Given equation becomes $\dfrac{dY}{dX} = \dfrac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$. ...(2)

Choose h, k so that $h+2k-3=0$ and $2h+k-3=0$. ...(3)

Solving (3) we get $h=1, k=1$ so that from (1), we have

$$X = x-1, Y = y-1. \quad ...(4)$$

Using (3) in (2), we get

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} = \frac{1+(2Y/X)}{2+(Y/X)}. \quad ...(5)$$

Take $\dfrac{Y}{X} = v$, *i.e.* $Y = vX$. So $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$. ...(6)

From (5) and (6), $v + X\dfrac{dv}{dX} = \dfrac{1+2v}{2+v}$ or $X\dfrac{dv}{dX} = \dfrac{1+2v}{2+v} - v = \dfrac{1-v^2}{2+v}$

or $$\frac{dX}{X} = \frac{(2+v)\,dv}{(1-v)(1+v)} = \left[\frac{1}{2}\left(\frac{1}{1+v}\right) + \frac{3}{2}\left(\frac{1}{1-v}\right)\right]dv.$$

[Resolving into partial fractions]

Integrating, $\log X + \log c = (1/2)\ [\log(1+v) - 3\log(1-v)]$

or $$2\log(cX) = \log\frac{1+v}{(1+v)^3} \quad \text{or} \quad X^2c^2 = \frac{1+v}{(1-v)^3}$$

or $$X^2c^2\left(1-\frac{Y}{X}\right)^3 = 1+\frac{Y}{X}, \quad \text{as } v = \frac{Y}{X}$$

or $$c^2(X-Y)^3 = X+Y \quad \text{or} \quad c^2\{x-1-(y-1)\}^2 = x-1+y-1, \text{ by (4)}$$

or $$c'(x-y)^2 = x+y-2, \text{ taking } c' = c^2.$$

Ex. 2. *Solve* $dy/dx + (x-y-2)/(x-2y-3) = 0.$ **(Ravishankar 93)**

Sol. Given equation is $dy/dx = -(x-y-2)/(x-2y-3)$.

Taking $x = X+h,\ y = Y+k$ so that $dy/dx = dY/dX$, ...(1)

The given equation becomes $\dfrac{dY}{dX} = -\dfrac{X-Y+h-k-2}{X-2Y+h-2k-3}$. ...(2)

Choose $h,\ k$ so that $h-k-2=0$ and $h-2k-3=0$. ...(3)

Solving (3), we get $h=1,\ k=-1$ so that from (1), we have

$$X = x-1 \text{ and } Y = y+1. \quad ...(4)$$

and (2) becomes $\dfrac{dY}{dX} = -\dfrac{X-Y}{X-2Y} = -\dfrac{1-(Y/X)}{1-2(Y/X)}$. ...(5)

Take $\dfrac{Y}{X} = v$, *i.e.* $Y = vX$. So $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$. ...(6)

From (5) and (6), $v + X\dfrac{dv}{dX} = -\dfrac{1-v}{1-2v}$ or $X\dfrac{dv}{dX} = \dfrac{1-2v^2}{2v-1}dv$

or $$\frac{dX}{X} = \frac{2v-1}{1-2v^2}dv \quad \text{or} \quad \frac{dX}{X} = \left[-\frac{1}{2}\frac{(-4v)}{1-2v^2} - \frac{1}{1-(v\sqrt{2})^2}\right]dv.$$

Integrating, $\log X = -\dfrac{1}{2}\log(1-2v^2) - \dfrac{1}{2\sqrt{2}}\log\dfrac{1+v\sqrt{2}}{1-v\sqrt{2}} - \dfrac{1}{2}\log c$

or $$2\log X + \log(1-2v^2) + \log c = -\frac{1}{\sqrt{2}}\log\left(\frac{1+v\sqrt{2}}{1-v\sqrt{2}}\right)$$

or $$\log\{cX^2(1-2v^2)\} = \log\left(\frac{1-v\sqrt{2}}{1+v\sqrt{2}}\right)^{1/\sqrt{2}}$$

or $$cX^2\left(1-2\frac{Y^2}{X^2}\right) = \left\{\frac{1-(Y/X)\sqrt{2}}{1+(Y/X)\sqrt{2}}\right\}^{1/\sqrt{2}} \quad \text{or} \quad c(X^2-2Y^2) = \left(\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}}\right)^{1/\sqrt{2}}$$

or $$c\{(x-1)^2 - 2(y+1)^2\} = \left\{\frac{x-1-(y+1)\sqrt{2}}{x-1+(y+1)\sqrt{2}}\right\}^{1/\sqrt{2}}$$

or $$c(x^2-2y^2-2x-4y-1) = \left(\frac{x-y\sqrt{2}-\sqrt{2}-1}{x+y\sqrt{2}-1+\sqrt{2}}\right)^{1/\sqrt{2}}$$

Ex. 3. *Solve* $(2x^2+3y^2-7)\,x\,dx - (3x^2+2y^2-8)\,y\,dy = 0$ **(I.A.S. 1995)**

2. $dy/dx = (y - x - 1)/(y + x + 5)$. **[Delhi B.Sc. (H) 93]**

Ans. $\log (x^2 + y^2 + 4x + 6y + 13) + 2 \tan^{-1} \{(y + 3)/(x + 2)\} = c$

3. $dy/dx = (2x + 2y - 2)/(3x + y - 5)$. **Ans.** $(y - x + 3)^4 = c\,(2x + y - 3)$

4. $dy/dx = (2x - y + 1)/(x + 2y - 3)$.

Ans. $(5y - 7)^2 + (5x - 1)(5y - 7) - (5x - 1)^2 = c$

5. $(x + 2y - 2)\, dx + (2x - y + 3)\, dy = 0$. **Ans.** $x^2 + 4xy - y^2 - 4x + 6y = c$

6. $(2x + 3y - 5)(dy/dx) + (3x + 2y - 5) = 0$. **Ans.** $3x^2 + 4xy + 3y^2 - 10x - 10y = c$

7. $(x - y)\, dy = (x + y + 1)\, dx$.

Ans. $\log \{c(x^2 + y^2 + x + y + 1/2)\} = 2 \tan^{-1} \{(2y + 1)/(2x + 1)\}$

8. $(6x + 2y - 10)(dy/dx) - 2x - 9y + 20 = 0$. **Ans.** $(y - 2x)^2 = c\,(x + 2y - 5)$

9. $(6x - 2y - 7)\, dx = (2x + 3y - 6)\, dy$.

Ans. $3y^2 + 4xy - 6x^2 + 14x - 12y - (9/2) = 0$

10. $(3y - 7x + 7)\, dx + (7y - 3x + 3)\, dy = 0$. **[Delhi B.Sc. (H) 95]**

Ans. $(y - x + 1)^2 (y + x - 1)^5 = c$

11. $(x - y - 1)\, dx + (4y + x - 1)\, dy = 0$.

Ans. $\log \{4y^2 + (x - 1)^2\} + \tan^{-1} \{2y/(x - 1)\} = c$

12. $(2x + 3y + 4)\, dy = (x + 2y + 3)\, dx$.

Ans. $\{(x - 1) + (y + 2)\sqrt{3}\}^{2 - \sqrt{3}} = c\, \{(x - 1) - (y + 2)\sqrt{3}\}^{2 + \sqrt{3}}$

13. Show that if the function $1/\{t - f(t)\}$ can be integrated (*w.r.t.* 't'), then one can solve $dy/dx = f(y/x)$, for any given f. Hence or otherwise solve

$(dy/dx) + (x - 3y + 2)/(3x - y + 6) = 0$ **(I.A.S. 1990)**