3.4.5 Divergence Criteria *If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.*

(i) *X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.*
(ii) *X is unbounded.*

3.4.6 Examples **(a)** The sequence $X := ((-1)^n)$ is divergent.

The subsequence $X' := ((-1)^{2n}) = (1, 1, \ldots)$ converges to 1, and the subsequence $X'' := ((-1)^{2n-1}) = (-1, -1, \ldots)$ converges to -1. Therefore, we conclude from Theorem 3.4.5(i) that X is divergent.

(b) The sequence $(1, \frac{1}{2}, 3, \frac{1}{4}, \ldots)$ is divergent.

This is the sequence $Y = (y_n)$, where $y_n = n$ if n is odd, and $y_n = 1/n$ if n is even. It can easily be seen that Y is not bounded. Hence, by Theorem 3.4.5(ii), the sequence is divergent.

(c) The sequence $S := (\sin n)$ is divergent.

This sequence is not so easy to handle. In discussing it we must, of course, make use of elementary properties of the sine function. We recall that $\sin(\pi/6) = \frac{1}{2} = \sin(5\pi/6)$ and that $\sin x > \frac{1}{2}$ for x in the interval $I_1 := (\pi/6, 5\pi/6)$. Since the length of I_1 is $5\pi/6 - \pi/6 = 2\pi/3 > 2$, there are at least two natural numbers lying inside I_1; we let n_1 be the first such number. Similarly, for each $k \in \mathbb{N}$, $\sin x > \frac{1}{2}$ for x in the interval

$$I_k := \big(\pi/6 + 2\pi(k-1), 5\pi/6 + 2\pi(k-1)\big).$$

Since the length of I_k is greater than 2, there are at least two natural numbers lying inside I_k; we let n_k be the first one. The subsequence $S' := (\sin n_k)$ of S obtained in this way has the property that all of its values lie in the interval $[\frac{1}{2}, 1]$.

Similarly, if $k \in \mathbb{N}$ and J_k is the interval

$$J_k := \big(7\pi/6 + 2\pi(k-1),\ 11\pi/6 + 2\pi(k-1)\big),$$

then it is seen that $\sin x < -\frac{1}{2}$ for all $x \in J_k$ and the length of J_k is greater than 2. Let m_k be the first natural number lying in J_k. Then the subsequence $S'' := (\sin m_k)$ of S has the property that all of its values lie in the interval $[-1, -\frac{1}{2}]$.

Given any real number c, it is readily seen that at least one of the subsequences S' and S'' lies entirely outside of the $\frac{1}{2}$-neighborhood of c. Therefore c cannot be a limit of S. Since $c \in \mathbb{R}$ is arbitrary, we deduce that S is divergent. □

The Existence of Monotone Subsequences

While not every sequence is a monotone sequence, we will now show that every sequence has a monotone subsequence.

3.4.7 Monotone Subsequence Theorem *If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.*

Proof. For the purpose of this proof, we will say that the mth term x_m is a "peak" if $x_m \geq x_n$ for all n such that $n \geq m$. (That is, x_m is never exceeded by any term that follows it

Note. ∞ is not considered here as a real number because we are not dealing with the extended real number system. In the latter case the definitions need lot of modifications.

ILLUSTRATIONS

1. $\{1+(-1)^n\}$ oscillates finitely.
2. $\left\{(-1)^n\left(1+\frac{1}{n}\right)\right\}$ oscillates finitely.
3. $\{n^2\}$ diverges to $+\infty$.
4. $\{-2^n\}$ diverges to $-\infty$.
5. $\{n(-1)^n\}$ oscillates infinitely.
6. $\left\{\frac{(-1)^{n-1}}{n!}\right\}$ converges to the limit 0.
7. $\left\{1+\frac{1}{n}\right\}$ converges to the limit 1.
8. $\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, \ldots\}$ is bounded below but unbounded above, and has a limit point 0 besides $+\infty$,

$$\underline{\lim}\ S_n = 0, \qquad \overline{\lim}\ S_n = +\infty.$$

The sequence oscillates infinitely.

9. $\{1, 2, 3, 2, 5, 2, 7, 2, 3, 2, 11, 2, 13, \ldots\}$,

where, $S_n = \begin{cases} 2, \text{ when } n \text{ is even,} \\ \text{lowest prime factor } (\neq 1) \text{ of } n, \text{ when } n \text{ is odd,} \end{cases}$

is bounded on the left but not on the right. It has infinite number of limit points 2, 3, 5, 7, 11, ..., so that

$$\underline{\lim}\ S_n = 2, \qquad \overline{\lim}\ S_n = +\infty.$$

The sequence oscillates infinitely.

10. The sequence $\left\{m+\frac{1}{n}\right\}$ where m, n are natural numbers, also oscillates infinitely, 1, 2, 3, ... being its limit points.

SOLVED EXAMPLES