Rule III. *If the equation* $Mdx + Ndy = 0$ *is of the form* $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$, *then* $1/(Mx - Ny)$ *is an integrating factor of provided* $(Mx - Ny) \neq 0$. **(I.A.S. 1991)**

Proof. Suppose that $$Mdx + Ndy = 0 \qquad \ldots(1)$$
is of the form $$f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0. \qquad \ldots(2)$$

Comparing (1) and (2), we have
$$\frac{M}{y f_1(xy)} = \frac{N}{x f_2(xy)} = \mu, \text{ (say)}$$
$$\Rightarrow \quad M = \mu y f_1(xy) \quad \text{and} \quad N = \mu x f_2(xy). \qquad \ldots(3)$$

Rewriting $Mdx + Ndy$, we have
$$Mdx + Ndy = \frac{1}{2}\left\{(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$
$$\Rightarrow \quad \frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left\{\frac{Mx + Ny}{Mx - Ny}\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$
$$= \frac{1}{2}\left\{\frac{f_1(xy) + f_2(xy)}{f_1(xy) - f_2(xy)}\, d(\log xy) + d\left(\log \frac{x}{y}\right)\right\}, \text{ by (3)}$$
$$= \frac{1}{2}\left\{f(xy)\, d(\log xy) + d\left(\log \frac{x}{y}\right)\right\}$$
$$\left[\text{on assuming } \frac{f_1(xy) + f_2(xy)}{f_1(xy) - f_2(xy)} = f(xy)\right]$$
$$= \frac{1}{2}\left\{f(e^{\log xy})\, d(\log xy) + d\left(\log \frac{x}{y}\right)\right\}$$
$$= \frac{1}{2}\left\{g(\log xy)\, d(\log xy) + d \log\left(\frac{x}{y}\right)\right\}$$
$$[\text{on assuming } f(e^{\log xy}) = g(\log xy)]$$
$$= d\left\{\frac{1}{2}\log\frac{x}{y} + \frac{1}{2}\int g(\log xy)\, d(\log xy)\right\},$$
showing that $Mx - Ny$ is an I.F. of $Mdx + Ndy = 0$.

Illustrative solved example of Type 8 based on Rule III

Ex. *Solve* $(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0$.

(Allahabad 90 ; Kanpur 94 ; Lucknow 93. 97 : Rajasthan 96)

Sol. Given $(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0 \ldots(1)$

Comparing (1) with $Mdx + Ndy = 0$, we have
$$M = y(xy \sin xy + \cos xy) \quad \text{and} \quad N = x(xy \sin xy - \cos xy), \qquad \ldots(2)$$
showing that (1) is of the form $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$.

Again, $Mx - Ny = xy(xy \sin xy + \cos xy) - xy(xy \sin xy - \cos xy)$

or $\quad Mx - Ny = 2xy \cos xy \neq 0,$

showing that I.F. of (1) $= 1/(Mx - Ny) = 1/(2xy \cos xy)$.

On multiplying (1) by $1/(2xy\cos xy)$, we have

$$(1/2)(y\tan xy + 1/x)\,dx + (1/2)\,(x\tan xy - 1/y)\,dy = 0$$

which must be exact and so by the usual rule, the required solution is

$$(1/2)\,(\log\sec xy + \log x) - (1/2)\log y = (1/2)\log c$$

or $\quad \log\sec xy + \log(x/y) = \log c \quad$ or $\quad (x/y)\sec xy = c.$

Exercise 2(H)

Solve the following differential equations :

1. $(x^3y^3 + x^2y^2 + xy + 1)\,y\,dx + (x^3y^3 - x^2y^2 - xy + 1)\,x\,dy = 0.$
   Ans. $x^2y^2 - 1 - 2xy\log y = 2cxy$

2. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0.$
   Ans. $xy - (1/xy) + \log(x/y) = c$

3. $(x^4y^4 + x^2y^2 + xy)\,y\,dx + (x^4y^4 - x^2y^2 + xy)\,x\,dy = 0.$
   Ans. $(1/2)\,x^2y^2 - (1/xy) + \log(x/y) = c$

4. (*a*) $y(1 - xy)\,dx - x(1 + xy)\,dy = 0.$ **(Agra 94 ; I.A.S. 69)**
   (*b*) $y(1 + xy)\,dx + x(1 - xy)\,dy = 0.$ **(Meerut 93 ; I.A.S. 92)**
   Ans. (*a*) $\log(x/y) - xy = c$ (*b*) $\log(x/y) - (1/xy) = c$

5. $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0.$ **Ans.** $\log(x^2/y) - (1/xy) = c$

Rule IV. *If* $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ *is a function of x alone say f (x), then* $e^{\int f(x)\,dx}$ *is an integrating factor of* $Mdx + Ndy = 0$.

(Calicut 93 ; Mysore 91 ; I.A.S. 77, 94)

Proof. Given equation is $\qquad Mdx + Ndy = 0 \qquad$ ...(1)

and $\quad \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(x) \quad$ so that $\quad Nf(x) = \dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}. \qquad$ ...(2)

Multiplying both sides of (1) by $e^{\int f(x)\,dx}$, we have

$$M_1\,dx + N_1\,dy = 0, \qquad ...(3)$$

where $\quad M_1 = M\,e^{\int f(x)\,dx}$ and $N_1 = N\,e^{\int f(x)\,dx} \qquad$ ...(4)

From (4), $\quad \dfrac{\partial M_1}{\partial y} = \dfrac{\partial M}{\partial y}\,e^{\int f(x)\,dx} \qquad$ ...(6)

and
$$\frac{\partial N_1}{\partial x} = \frac{\partial N}{\partial x}\,e^{\int f(x)\,dx} + N\,e^{\int f(x)\,dx} f(x)$$
$$= e^{\int f(x)\,dx}\left\{\frac{\partial N}{\partial x} + Nf(x)\right\}$$
$$= e^{\int f(x)\,dx}\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right), \text{ by (2)}$$

so that $\quad \dfrac{\partial N_1}{\partial x} = e^{\int f(x)\,dx}\,\dfrac{\partial M}{\partial y}. \qquad$ ...(7)

$\therefore$ From (6) and (7), $\partial M_1/\partial y = \partial N_1/\partial x$,

showing that $M_1 dx + N_1 dy = 0$ must be exact and hence $e^{\int f(x)\,dx}$ is an I.F. of (1) as desired.

Illustrative solved example of Type 9 based on Rule IV

Ex. *Solve* $(x^2 + y^2 + x)\,dx + xy\,dy = 0.$ **(Delhi Hons. 97)**

Sol. Given $(x^2 + y^2 + x)\,dx + xy\,dy = 0.$ ...(1)

Comparing (1) with $Mdx + Ndy = 0$, we have

$$M = x^2 + y^2 + x \quad \text{and} \quad N = xy. \qquad ...(2)$$

Here $\partial M/\partial y = 2y$ and $\partial N/\partial x = y.$ ...(3)

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - y) = \frac{1}{x},$$

which is a function of x alone.

$\therefore$ I.F. of (1) $= e^{\int (1/x)\,dx} = e^{\log x} = x.$

Multiplying (1) by x, we have $(x^3 + xy^2 + x^2)\,dx + x^2 y\,dy = 0,$ which must be exact equation and so its solution is

$$(x^4/4) + (x^2 y^2)/2 + (x^3/3) = c/12 \quad \text{or} \quad 3x^4 + 6x^2y^2 + 4x^3 = c.$$

Exercise 2(I)

Solve the following differential equations :

1. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0.$ **Ans.** $e^x(x^2 + y^2) = c$
2. $(x^3 - 2y^2)\,dx + 2xy\,dy = 0.$ **Ans.** $x + (y^2/x^2) = c$
3. $(y + y^3/3 + x^2/2)\,dx + (1/4)(x + xy^2)\,dy = 0.$ **Ans.** $x^6 + 3x^4y + x^4y^3 = c$
   (Allahabad 94 ; Kanpur 87 ; Lucknow 92)
4. $(x^2 + y^2)\,dx - 2xy\,dy = 0.$ **Ans.** $x^2 - y^2 = cx$
5. $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0.$ **Ans.** $x^2 - 1 - y^2 = cx$
6. $(x^2 + y^2 + 1)\,dx + x(x - 2y)\,dy = 0.$ **Ans.** $x + y - (y^2 + 1)/x = c$

Rule V. *If* $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ *is a function of y alone say* $f(y)$, *then* $e^{\int f(y)\,dy}$ *is an integrating factor of* $Mdx + Ndy = 0.$

Proof. Proceed exactly as for Rule IV.

Illustrative solved example of Type 10 based on Rule V

Example. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0.$ ...(1)

Sol. Comparing (1) with $Mdx + Ndy = 0$, we get

$M = 2xy^4e^y + 2xy^3 + y$ and $N = x^2y^4e^y - x^2y^2 - 3x.$ ...(2)

Here $\partial M/\partial y = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$, $\partial N/\partial x = 2xy^4e^y - 2xy^2 - 3.$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4(2xy^3e^y + 2xy^2 + 1) = -\frac{4}{y}(2xy^4e^y + 2xy^3 + y) = \frac{-4M}{y}$$

$$\Rightarrow \quad \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=-\frac{4}{y}, \text{ which is a function of } y \text{ alone.}$$

$\Rightarrow$ I.F. of (1) $= e^{\int(-4/y)\,dy} = e^{-4\log y} = (1/y^4)$.

Multiplying (1) by $1/y^4$, we have

$$\{2xe^y + (2x/y) + (1/y^3)\}\,dx + \{x^2e^y - (x^2/y^2) - 3\,(x/y^4)\}\,dy = 0,$$

which must be exact and so by usual method its solution is

$$x^2 e^y + (x^2/y) + x/y^3 = c.$$

Exercise 2(J)

Solve the following differential equations :

1. $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0.$ **(Mysore 92)**
   Ans. $e^y\,[(x^2y^2)/2 - (1/3)\,x^3 + (1/6)(y^2 - y/3 + 1/18)] = c$
2. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0.$ **Ans.** $3x^2y^4 + 6xy^2 + 2y = c$
3. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0.$ **(Gorakhpur 92)** **Ans.** $x^3y^3 + x^2 = cy$
4. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$ **(Delhi 93)** **Ans.** $x\,\{y + (2/y^2)\} + y^2 = c$

Rule VI. *If the given equation* $Mdx + Ndy = 0$, *is of the form* $x^\alpha y^\beta\,(my\,dx + nx\,dy) = 0$, *then its integrating factor is* $x^{km-1-\alpha}\,y^{kn-1-\beta}$, *where k can have any value.*

Proof. By assumption, the given equation can be written as

$$x^\alpha y^\beta\,(my\,dx + nx\,dy) = 0. \qquad \ldots(1)$$

Multiplying (1) by $x^{km-1-\alpha}\,y^{kn-1-\beta}$, we have

$$x^{km-1}\,y^{kn-1}\,(my\,dx + nx\,dy) = 0$$

or $\quad km\,x^{km-1}\,y^{kn}\,dx + kn\,y^{kn-1}\,x^{km}\,dy = 0 \;\Rightarrow\; d\,(x^{km}y^{kn}) = 0,$

showing that $x^{km-1-\alpha}\,y^{kn-1-\beta}$ is an I.F. of the given equation (1).

Remark 1. Using rule VI, we now find the rule for finding an I.F. of the equation of the form

$$x^\alpha y^\beta\,(my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'}\,(m'\,y dx + n'\,x\,dy) = 0 \qquad \ldots(2)$$

By virtue of rule VI, we see that the factor that makes the first term of (2) exact differential is $x^{km-1-\alpha}\,y^{kn-1-\beta}$ and that for the second term of (2) is $x^{k'm'-1-\alpha'}\,y^{k'n'-1-\beta'}$ where k and k' can have any value.

The above mentioned two factors will be identical if we choose k and k' such that

$$km - 1 - \alpha = k'm' - 1 - \alpha' \qquad \ldots(3)$$

and

$$kn - 1 - \beta = k'n' - 1 - \beta'. \qquad \ldots(4)$$

Solving (3) and (4), we evaluate the values of k and k'. Substituting these values in the factor

$$x^{km-1-\alpha}\,y^{kn-1-\beta} \quad \text{or} \quad x^{k'm'-1-\alpha'}\,y^{k'n'-1-\beta'},$$

we obtain the required I.F. of (2).

Illustrative solved examples of Type 11 based on rule VI

Ex. *Solve* $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0.$ ...(1)

Sol. Rewriting (1) in the standard form

$$x^{\alpha} y^{\beta} (my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'} (m'y\,dx + n'x\,dy) = 0, \quad ...(2)$$

we have $y\,(y\,dx - x\,dy) + x^2\,(2y\,dx + 2x\,dy) = 0.$ ...(3)

Comparing (2) and (3), we have

$$\alpha = 0, \beta = 1, m = 1, n = -1; \alpha' = 2, \beta' = 0, m' = 2, n' = 2.$$

Hence the I.F. for the first term on L.H.S. of (3) is

$$x^{k-1} y^{-k-1-1}, \text{ i.e., } x^{k-1} y^{-k-2} \quad ...(4)$$

and the I.F. for the second term on L.H.S. of (3) is

$$2^{2k'-1-2} y^{2k'-1}, \text{ i.e., } x^{2k'-3} y^{2k'-1}. \quad ...(5)$$

For the integrating factors (4) and (5) to be identical, we have

$$k - 1 = 2k' - 3 \text{ and } -k - 2 = 2k' - 1$$

$$\Rightarrow k - 2k' = -2 \text{ and } k + 2k' = -1 \Rightarrow k = -3/2 \text{ and } k' = 1/4 \quad ...(6)$$

Substituting the value of k in (4) or k' in (5), the integrating factor of (3) or (1) is $x^{-5/2} y^{-1/2}$.

Multiplying (1) by $x^{-5/2} y^{-1/2}$, we have

$$(x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2})\,dx + (2x^{1/2} y^{-1/2} - x^{-3/2} y^{1/2})\,dy = 0,$$

which must be exact and so by the usual rule its solution is given by

$$\frac{x^{-3/2} y^{3/2}}{(-3/2)} + \frac{2x^{1/2} y^{1/2}}{(1/2)} = \frac{2C}{3} \quad \text{or} \quad 6x^{1/2} y^{1/2} - x^{-3/2} y^{3/2} = C.$$

Remark 2. Sometimes the Rule VI for finding I.F. is modified as given below:

If the given equation $M dx + N dy = 0$ can be put in the form

$$x^{\alpha} y^{\beta} (my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'} (m'y\,dx + n'x\,dy) = 0,$$

where $\alpha, \beta, m, n, \alpha', \beta', m', n'$ are constants, then the given equation has an I.F. $x^h y^k$, where h and k are obtained by applying the condition that the given equation must become exact after multiplying by $x^h y^k$.

Illustrative solved examples based on Remark 2

Ex. 1. *Solve* $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0.$

(Kanpur 92 ; Allahabad 93 ; Meerut 84 : Lucknow 93 ; Baroda 89)

Sol. The given equation can be rewritten as

$$(y^2\,dx - xy\,dy) + (2x^2y\,dx + 2x^3\,dy) = 0$$

or

$$y\,(y\,dx - x\,dy) + x^2\,(2y\,dx + 2x\,dy) = 0.$$

So let $x^h y^k$ be an I.F. of the given equation.

Multiplying the given equation by $x^h y^k$, we have

$$(x^h y^{k+2} + 2x^{h+2} y^{k+1})\,dx + (2x^{h+3} y^k - x^{h+1} y^{k+1})\,dy = 0, \quad \ldots(1)$$

which must be exact. For (1), we have

$$M = x^h y^{k+2} + 2x^{h+2} y^{k+1},\ N = 2x^{h+3} y^k - x^{h+1} y^{k+1}.$$

Since (1) is exact, we must have $\partial M/\partial y = \partial N/\partial x$

i.e., $(k+2)\,x^h y^{k+1} + 2\,(k+1)\,x^{h+2} y^k = 2\,(h+3)\,x^{h+2} y^k - (h+1)\,x^h y^{k+1}$

Now equating the coefficients of $x^h y^{k+1}$ and $x^{h+2} y^k$, we get

$$k+2 = -(h+1) \text{ and } 2\,(k+1) = 2\,(h+3),$$

i.e., $h + k = -3$ and $h - k = -2.$ ...(2)

Solving system (2), we get $h = -(5/2)$ and $k = -(1/2)$

$$\therefore \quad \text{I.F.} = x^{-5/2} y^{-1/2}.$$

Multiplying the given equation by I.F. $x^{-5/2} y^{-1/2}$, we get

$$(x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2})\,dx + (2x^{1/2} y^{-1/2} + x^{-3/2} y^{1/2})\,dy = 0$$

which must be exact. For this new equation, as usual solution is

$$-(2/3)\,x^{-3/2} y^{3/2} + 4x^{1/2} y^{1/2} = c.$$

Ex. 2. *Given that the differential equation* $(2x^2y^2 + y)\,dx - (x^3y - 3x)dy = 0$ *has an I.F. of the form* $x^h y^k$, *find its general solution.* **(Kakitiya 97)**

Sol. Given $(2x^2y^2 + y)\,dx + (3x - x^3 y)\,dy = 0.$ ...(1)

Multiplying both sides of (1) by I.F. $x^h y^k$, we get

$$(2x^{h+2} y^{k+2} + x^h y^{k+1})\,dx + (3x^{h+1} y^k - x^{h+3} y^{k+1})\,dy = 0, \quad \ldots(2)$$

which must be exact. Comparing (2) with $Mdx + Ndy = 0$, we have

$$M = 2x^{h+2} y^{k+2} + x^h y^{k+1} \text{ and } N = 3x^{h+1} y^k - x^{h+3} y^{k+1}. \quad \ldots(3)$$

For (2) to be exact, $\partial M/\partial y = \partial N/\partial x$

$\Rightarrow$ $2\,(k+2)\,x^{h+2} y^{k+1} + (k+1)\,x^h y^k = 3\,(h+1)\,x^h y^k - (h+3)\,x^{h+2} y^{k+1}$

$\Rightarrow$ $2\,(k+2) = -(h+3)$ and $k + 1 = 3\,(h+1)$

$\Rightarrow$ $h + 2k = -7$ and $3h - k = -2$ $\Rightarrow$ $h = -11/7$ and $k = -19/7.$

So an I.F. of (1) is $x^{-11/7} y^{-19/7}$. Multiplying (1) by $x^{-11/7} y^{-19/7}$, we have

$$(2x^{3/7} y^{-5/7} + x^{-11/7} y^{-12/7})\,dx + (3x^{-4/7} y^{-19/7} - x^{10/7} y^{-12/7})\,dy = 0,$$

which must be exact. Hence, as usual, the required solution is

$$\frac{2x^{10/7} y^{-5/7}}{10/7} + \frac{x^{-4/7} y^{-12/7}}{(-4/7)} = \frac{7c}{20} \quad \text{or} \quad 4x^{10/7} y^{-5/7} - 5x^{-4/7} y^{-12/7} = c.$$

Exercise 2(K)

Solve the following differential equations :

1. $(2y\,dx + 3x\,dy) + 2xy\,(3y\,dx + 4x\,dy) = 0.$ **(Kanpur 97, 98)**

   Ans. $x^2 y^3 + 2x^3 y^4 = c$

2. $(2x^2 y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0.$

   Ans. $12x^{-10/13} y^{15/13} + 5x^{-36/13} y^{24/13} = c$

3. $(3x+2y^2)\,y\,dx+2x\,(2x+3y^2)\,dy=0.$ **Ans.** $x^3y^4+x^2y^6=c$

4. $x\,(3y\,dx+2x\,dy)+8y^4\,(y\,dx+3x\,dy)=0.$ **Ans.** $x^2y^3\,(x+4y^4)=c$

5. $x\,(4y\,dx+2x\,dy)+y^3\,(3y\,dx+5x\,dy)=0.$ **(Delhi B.Sc. 99) Ans.** $x^4y^3+x^3y^5=c$

6. $xy^3\,(y\,dx+2x\,dy)+(3y\,dx+5x\,dy)=0.$ **Ans.** $x^3y^5\,(xy^3+4)=c$

(Rohilkhand 92)

2·12. Linear differential equation

Definition. A first order differential equation is called linear if it can be written in the form
$$(dy/dx)+Py=Q, \qquad \text{...(1)}$$
where P and Q are constants or functions of x alone (*i.e.* not of y).

A method of solving (1). Suppose R (which is taken as function of x alone) is an integrating factor of (1). Multiplying (1) by R, we get
$$R\frac{dy}{dx}+RPy=RQ, \qquad \text{...(2)}$$
which must be exact. Suppose we wish that the L.H.S. of (2) is the differential coefficient of some product. But the term $R\,(dy/dx)$ can only be obtained by differentiating the product Ry. Accordingly, we take
$$R\frac{dy}{dx}+RPy=\frac{d}{dx}(Ry) \qquad \text{...(3)}$$
$$R\frac{dy}{dx}+RPy=R\frac{dy}{dx}+y\frac{dR}{dx} \quad \text{or} \quad \frac{dR}{R}=P\,dx.$$

Integrating, $\log R=\int P\,dx$ [take constant of integration equal to the zero for sake of simplicity].

Thus, an integrating factor of (1) is $R=e^{\int P\,dx}$ and (2) can be written as
$$\frac{d}{dx}(Ry)=RQ \qquad \text{[using (3)]}$$
or
$$d\,(Ry)=RQ\,dx.$$
Integrating,
$$Ry=\int RQ\,dx+c$$
i.e.,
$$ye^{\int P\,dx}=\int\{Qe^{\int P\,dx}\}\,dx+c,$$
which is the required solution of given linear differential equation (1).

Working rule for solving linear equations. First put the given equation in the standard form (1). Next find an integrating factor (I.F.) by using formula
$$\text{I.F.}=e^{\int P\,dx}. \qquad \text{...(5)}$$

Two formulas $e^{m\log A}=A^m$ and $e^{-m\log A}=1/A^m$ will be often used in simplifying I.F.

Lastly, the required solution is obtained by using the result
$$y\times(\text{I.F.})=\int[Q\times(\text{I.F.})]\,dx+c, \qquad \text{...(6)}$$
where c is an arbitrary constant.

Remarks. Sometimes a differential equation cannot be put in the form (1) of a linear equation. Then we regard y as the independent variable and x as the dependent variable and obtain a differential equation of the form

$$\frac{dx}{dy}+P_1x=Q_1, \quad ...(7)$$

where P_1 and Q_1 are constants or functions of y alone. In this, we modify the above working rule as follows.

$$\text{I.F.}=e^{\int P_1\,dy} \quad ...(8)$$

and the required solution is $x\times(\text{I.F.})=\int[Q_1\times(\text{I.F.})]\,dy+c.$

Examples of Type 12 based on Art. 2·12

Ex. 1. *Solve* $x\cos x\,(dy/dx)+y\,(x\sin x+\cos x)=1.$

(Agra 94 ; Lucknow 88 ; Meerut 87 ; U.P.P.C.S. 78)

Sol. Re-writing given equation, we have

$$\frac{dy}{dx}+\left(\tan x+\frac{1}{x}\right)y=\frac{\sec x}{x}$$

$$\text{I.F.}=e^{\int(\tan x+1/x)\,dx}=e^{\log\sec x+\log x}=e^{\log x\sec x}=x\sec x.$$

Hence the required solution is

$$yx\sec x=\int\sec^2 x\,dx+c \quad \text{or} \quad yx\sec x=\tan x+c$$

Ex. 2. *Solve* $(1-x^2)(dy/dx)+2xy=x\sqrt{(1-x^2)}.$ **(Kerala 2001)**

Sol. The given equation is $\dfrac{dy}{dx}+\dfrac{2x}{1-x^2}y=\dfrac{x}{\sqrt{(1-x^2}}.$

Here $\int P\,dx=\int\dfrac{2x}{1-x^2}\,dx=-\log(1-x^2)$ so $\text{I.F.}=e^{\int P\,dx}=\dfrac{1}{1-x^2}$

So the required solution is

$$\frac{y}{1-x^2}=\int\frac{x}{\sqrt{(1-x^2)}}\times\frac{1}{1-x^2}\,dx.$$

$$=-\frac{1}{2}\int t^{-3/2}\,dt+c \qquad [\text{Put } 1-x^2=t \;\therefore\; -2x\,dx=dt]$$

$$=t^{-1/2}+c=c+1/\sqrt{t}$$

or $$\frac{y}{1-x^2}=\frac{1}{\sqrt{(1-x^2)}}+c. \qquad [\because\; t=1-x^2]$$

Ex. 3. *Solve* $\sin x\,(dy/dx)+3y=\cos x.$ **(Agra 91 : Rohilkhand 93)**

Sol. Rewriting, we have, $\dfrac{dy}{dx}+(3\operatorname{cosec}x)\,y=\cot x.$

Here $\int P\,dx=3\int\operatorname{cosec}x\,dx=3\log\tan(x/2)$ so $\text{I.F.}=e^{\int P\,dx}=\tan^3 x/2$

So the required solution is

$$y \tan^3 (x/2) = \int \cot x \tan^3 (x/2)\, dx + c$$

or $$y \tan^3 \frac{x}{2} = \int \frac{1 - \tan^2 x/2}{2 \tan x/2} \tan^3 \frac{x}{2}\, dx + c$$

or $$y \tan^3 \frac{x}{2} = \frac{1}{2} \int \left(1 - \tan^2 \frac{x}{2}\right) \tan^2 \frac{x}{2}\, dx + c$$

or $$y \tan^3 \frac{x}{2} = \frac{1}{2} \int (1 - t^2)\, t^2 \times \frac{2\, dt}{1 + t^2} + c,$$

$$\left[\text{Put } \tan \frac{x}{2} = t \Rightarrow \tfrac{1}{2} \sec^2 \frac{x}{2}\, dx = dt \right.$$

$$\left. \Rightarrow dx = \frac{2\, dt}{\sec^2 x/2} = \frac{2\, dt}{1 + \tan^2 x/2} = \frac{2\, dt}{1 + t^2}\right]$$

or $$y \tan^3 \frac{x}{2} = \int \frac{t^2 - t^4}{1 + t^2}\, dt + c \quad \text{or} \quad y \tan^3 \frac{x}{2} = \int \left[-t^2 + 2 - \frac{2}{t^2 + 1}\right] dt$$

or $$y \tan^3 (x/2) = -(1/3)\, t^3 + 2t - 2 \tan^{-1} t + c$$

or $$y \tan^3 \frac{x}{2} = -\tfrac{1}{3} \tan^3 \frac{x}{2} + 2 \tan \frac{x}{2} - 2 \tan^{-1}\left(\tan \frac{x}{2}\right) + c$$

or $$(y + 1/3) \tan^3 (x/2) = 2 \tan (x/2) - x + c.$$

Ex. 4. *Integrate* $(1 + x^2)(dy/dx) + 2xy - 4x^2 = 0$. *Obtain equation of the curve satisfying this equation and passing through the origin.* **(Agra 93)**

Sol. Rewriting the given equation, $\dfrac{dy}{dx} + \dfrac{2x}{1 + x^2}\, y = \dfrac{4x^2}{1 + x^2}$.

Here $\int P\, dx = \int \dfrac{2x}{1 + x^2}\, dx = \log (1 + x^2)$ so I.F. $= e^{\int P\, dx} = (1 + x^2)$.

Hence the required solution is

$$y (1 + x^2) = \int \frac{4x^2}{1 + x^2} \cdot (1 + x^2)\, dx + c$$

or $$y (1 + x^2) = (4/3)\, x^3 + c. \qquad \text{...(1)}$$

Since the required curve passes through origin, (1) must satisfy the condition $x = 0, y = 0$. Putting these in (1), we get $c = 0$.

Hence the required curve is $4x^3 = 3y (1 + x^2)$.

Ex. 5. *Solve* $(x + 2y^3)(dy/dx) = y$. **(Rohilkhand 93 ; Agra 95 Delhi B.Sc. (G) 95, 2002 ; Lucknow 95 ; Ravishankar 92)**

Sol. Here it is possible to put the equation in form

$$dx/dy + P_1 x = Q_1.$$

Thus, we have $\dfrac{dx}{dy} = \dfrac{x + 2y^3}{y}$ i.e. $\dfrac{dx}{dy} - \dfrac{1}{y}\, x = 2y^2$. ...(1)

In (1), $\int P_1\, dy = -\int (1/y)\, dy = -\log y \quad \therefore \quad$ I.F. $= e^{-\log y} = 1/y$.

$$= c - \frac{a}{2}\left[\frac{t^{-1/2}}{-1/2}\right] = c + \frac{a}{\sqrt{t}} = c + \frac{a}{\sqrt{(x^2-1)}}$$

or $$y = cx\sqrt{(x^2-1)} + ax.$$

Exercise 2(L)

1. $(1+x^2)(dy/dx) + y = e^{\tan^{-1} x}$. **Ans.** $ye^{\tan^{-1} x} = (1/2)\, e^{2\tan^{-1} x} + c$
2. $(dy/dx) + y \cot x = 2\cos x$. **(Balgalore 94)** **Ans.** $y \sin x = -(1/2)\cos 2x + c$
3. $(dy/dx) + y \tan x - \sec x = 0$. **Ans.** $y \sec x = c + \tan x$
4. $dy/dx = y \tan x - 2\sin x$. **Ans.** $y = \cos x + c \sec x$
5. $(dy/dx) + 2y \tan x = \sin x$, given that $y = 0$ when $x = \pi/3$. **Ans.** $y = \cos x - 2\cos^2 x$
6. $\cos^2 x\,(dy/dx) + y = \tan x$. **(Bangalore 91)** Ans. $y = \tan x - 1 + c\,e^{-\tan^{-1} x}$
7. $dy/dx + 2\,(y/x) = \sin x$. **Ans.** $yx^2 = c - x^2\cos x + 2x \sin x + 2\cos x$
8. $(2x - 10y^3)(dy/dx) + y = 0$. **(Allahabad 93)** **Ans.** $xy^2 = c + 2y^5$
9. $(x \log x)(dy/dx) + y = 2\log x$. **(Delhi B.Sc. 96)** **Ans.** $y \log x = c + (\log x)^2$
10. $\cos x\,(dy/dx) + y = \sin x$ or $(dy/dx) + y \sec x = \tan x$. **Ans.** $y(\sec x + \tan x) = \sec x + \tan x - x + c$
11. $(1+y^2) + (x - e^{\tan^{-1} y})(dy/dx) = 0$. **Ans.** $xe^{\tan^{-1} y} = (1/2)\, e^{2\tan^{-1} y} + c$
12. $x\,(dy/dx) - y = 2x^2 \operatorname{cosec} x$. **(Kanpur 96)** **Ans.** $y = cx + x \log(\tan x)$
13. $x^2(x^2-1)(dy/dx) + x(x^2+1)\,y = x^2 - 1$. **(Lucknow 94 ; Rohilkhand 94)** **Ans.** $\{y(x^2-1)\}/x = \log x + (1/2)\,x^{-2} + c$
14. $x(x-1)(dy/dx) - (x-2)\,y = x^3(2x-1)$. **Ans.** $\{y(x-1)\}/x^2 = x^2 - x + c$
15. $(1+x^2)(dy/dx) + 2xy = \cos x$. **(Merrut 98)** **Ans.** $y(1+x^2) = c + \sin x$
16. $(dy/dx) - y \tan x = e^x \sec x$. **Ans.** $y \cos x = c + e^x$
17. $\sec x\,(dy/dx) = y + \sin x$. **Ans.** $y = ce^{\sin x} - (1 + \sin x)$
18. $y \log y\, dx + (x - \log y)\, dy = 0$. **(Delhi Hons. 1989)** **Ans.** $x \log y = (1/2)(\log y)^2 + c$
19. $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1}\,y = \dfrac{1}{(x^2+1)^2}$. **(Delhi Hons. 1995)** **Ans.** $y(x^2+1)^2 = x + c$
20. $\sin 2x\,(dy/dx) = y + \tan x$. **Ans.** $y = \tan x + c\sqrt{(\tan x)}$
21. $(x + 3y + 2)(dy/dx) = 1$. **Ans.** $x + 3y + 5 = ce^y$
22. $(1 - x^2)(dy/dx) - xy = 1$. **Ans.** $y\sqrt{(x^2-1)} = c - \log[(x + \sqrt{(x^2-1)})]$
23. $(dy/dx) + (y/x) = x^2$, if $y = 1$ when $x = 1$. **Ans.** $4xy = x^4 + 3$
24. $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}\,y = \dfrac{1}{(1+x^2)^2}$ if $y = 0$, when $x = 1$. **Ans.** $y(1+x^2) = \tan^{-1} x - (\pi/4)$