3 Matrix Algebra and Economic Application

Matrix algebra is a very powerful mathematical tool to deal with the problem of solving a system of simultaneous linear equationis. In most of the economic problems, the behaviour of economic variables are normally assumed to be linear although non-linear relations exist in practice. Even within the restrictive assumption of linearity, we are to find out the equilibrium values of the endogeneous variables in a system of linear simultaneous equations. For instance in chapter-II, we dealt with the problem of finding the equilibrium price and quantity in a simple market model of three simultaneous

$$\left.\begin{aligned} Q_d &= f(P) = \alpha_0 + \alpha_1 P \\ Q_s &= g(P) = \beta_0 + \beta_1 P \\ Q_d &= Q_s \end{aligned}\right\} \tag{3.1}$$

equations of the form with certain restrictions on the values of the parameters α_0, α_1, β_0 and β_1 to have economic sense. But when we extend the above single commodity market model to cover a market situation where two goods (may be complementary or competitive) exist, it is also realistic to assume that the demand for one commodity depends on the prices of both as well as the supply of one commodity depends on the prices of both. So we will have in such a market model, two demand functions, two supply functions and two equilibrium conditions such that

$$\begin{aligned} Q_{d_1} &= \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \\ Q_{s_1} &= \beta_0 + \beta_1 P_1 + \beta_2 P_2 \\ Q_{d_2} &= \gamma_0 + \gamma_1 P_1 + \gamma_2 P_2 \\ Q_{s_2} &= \delta_0 + \delta_1 P_1 + \delta_2 P_2 \\ Q_{d_1} &= Q_{s_1} \text{ and } Q_{d_2} = Q_{s_2} \end{aligned} \tag{3.2}$$

where P_1 and P_2 are prices of first and second commodity, $Q_{d_1}, Q_{s_1}, Q_{d_2}, Q_{s_2}$ are the demand and supply functions of first and second commodities respectively.

In the above market model (3.2) there are six linear equations and six variables and so we get the feasible solution of the system. Likewise, if we have a market model of three related goods (either supplementary or competitive), there will be nine equations and nine variables. In general terms, if we have a market model of n goods, we will have $3n$ equations and $3n$ variables. When we have two/three equations or a limited number of equations, we can solve the system of equations by doing simple mathematical deductions. But when we have a large number of equations (say, n equations), it is extremely difficult to solve the system of equatioins by ordinary mathematical deduction. In such a situatioin, several important questions arise if we wish to analyse a multi-good linear market model.

1. Can we write a large number of equations with a large number of variables in a compact way?
2. Whether there is an unique solution to such a linear simultaneous equations system?
3. If solution exists, how do we solve the system of equations to find equilibrium prices and quantities.

All the answers to the above three questions are found in matrix algebra. Matrix algebra enables us (a) to write a system of equations in compact form, (b) to test the existence of unique solution and (c) to solve for equilibrium prices and quantities.

3.1 MATRICES AND VECTORS DEFINED

In order to explain the concept of matrix, we define a system of linear simultaneous equations with two equations and two variables such that

$$\begin{aligned} 2x_1 + 3x_2 &= 8 \\ 4x_1 + 2x_2 &= 8. \end{aligned} \tag{3.3}$$

In general form, the above two simultaneous equations are written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= C_1 \\ a_{21}x_1 + a_{22}x_2 &= C_2. \end{aligned} \tag{3.4}$$

The double-digit subscript of the coefficient 'a' attached to each variable (x_1 or x_2) has its own interpretation. The first subscript refers to the equation number and second subscript refers to the variable. For example, a_{12} is the coefficient of the variable x_2 in the first equation. Thus in general a_{ij} refers to the coefficient of x_j in the ith equation. Similarly, C_i refers to the constant term of the ith equation. The linear two equations system (3.4) can easily be extended to a linear n-equation system with m variables such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m &= C_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m &= C_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m &= C_n \end{aligned} \tag{3.5}$$

where the subscripts of (3.5) have the same interpretation as already stated in case of (3.4).

Since in the system of equations (3.5) we have a large number of variables (m) and a large number of equations (n), we are to see how to express them in a compact form. Both the system of equations (3.4) and (3.5) consist of three component values—(a) the set of coefficients (a_{ij}), (b) the set of variables $x_1, x_2, \ldots x_m$ and (c) the set of constants $C_1, C_2, \ldots C_n$. All these three components can be expressed in rectangular arrays of number symbolising as A, X and C respectively. For the set of equations (3.4).

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{3.6}$$

Similarly for the system of equation (3.5)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \tag{3.7}$$

If we define a matrix as a rectangular array of number, parameters or variables arranged in rows and columns, then each of the three rectangular arrays in (3.6) or in (3.7), is a matrix.

If we go back to the single commodity market model (3.1), all the three equations taken together can be converted into three rectangular arrays as shown in (3.6) and (3.7). We can rewrite the system of equation (3.1) as

$$\begin{aligned} 1\times Q_d + 0\times Q_s - \alpha_1 P &= \alpha_0 \\ 0\times Q_d + 1\times Q_s - \beta_1 P &= \beta_0 \\ 1\times Q_d - 1\times Q_s + 0\times P &= 0 \end{aligned} \tag{3.8}$$

Like (3.6) and (3.7), we can write

$$A = \begin{bmatrix} 1 & 0 & -\alpha_1 \\ 0 & 1 & -\beta_1 \\ 1 & -1 & 0 \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \quad C = \begin{bmatrix} \alpha_0 \\ \beta_0 \\ 0 \end{bmatrix} \tag{3.9}$$

It will be seen that all the three sets of simultaneous equations (3.4), (3.5) and (3.8) can be expressed in compact form

$$AX = C. \tag{3.10}$$

The detailed procedure of expressing the system of equations in matrix form will be discussed while dealing with matrix multiplication. The only point is to be noted that the number of values in A, X and C varies from each set of simultaneous equations (3.4), (3.5) and (3.8) referred above.

Each member of the array in a matrix is called "the element of a matrix". For instance, $a_{11}, a_{12}, \ldots, a_{nm}$ are the elements of matrix A or $x_1, x_2, \ldots, x_m$ are the elements of matrix X or $c_1, c_2, \ldots, c_n$ are the elements of matrix C when we consider the simultaneous equation system (3.5). The elements of a matrix A can also be expressed as

$$A = [a_{ij}] \quad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m \end{matrix}$$

The value of i and j indicate the location of an element in a matrix. For instance if $a_{ij} = a_{23}$, the element is located at the point of intersection of second row and third column of the matrix A.

The number of rows and columns of a matrix determined the 'dimension or order' of the matrix. For instance, matrix A has two rows and two columns in (3.6) and so the dimension of A is (2×2). Similarly dimension of A in (3.7) is $(n \times m)$ as there are n rows and m columns of A. The matrix X in (3.7) has only one column and each element constitutes a row. It has m rows. So the dimension of X is $(m \times 1)$. Similarly, the dimension of C is $(n \times 1)$. While writing the dimensions, it should be noted that the first value within bracket should be the number of rows and second value should be the number of columns. That is why the dimensions of A in (3.7) is written as $(n \times m)$ but not $(m \times n)$.

A matrix which has only one column or only one row, is known as a 'vector'. A matrix having only one column is called 'column vector' while a matrix having only one row is called a 'row vector'. For instance, the X matrix in (3.6) or in (3.7) is a column vector since we have only one column. But the dimension of the column vector X in (3.6) is (3×1) and that in (3.7) is $(m \times 1)$.

A matrix which has only one element, that is, having one row and one column is known as a 'scalar'. The scalar almost behaves like a number in ordinary algebra. The detailed use of scalar in matrix operation will be dealt in section 3.2.