3.4 THE TRANSPOSE OF A MATRIX

The transpose of a matrix A, denoted by A' is obtained by interchanging the rows and columns of A such that if A is a matrix of order $(m \times n)$, then A' will be a matrix of order $(n \times m)$.

For example, if $A = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix}_{2\times3}$, then the transpose of A, that is A' will be a matrix of order (3×2)

$$A' = \begin{bmatrix} 2 & 4 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}_{3\times2}$$

But in case of symmetric matrix, the transpose of the matrix (A') is equal to the original matrix (A). For example, two symmetric matrices C and D are given below and their transposes are equal to C and D respectively.

$$C = \begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix} = C'' \quad D = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 5 \\ 4 & 5 & 3 \end{bmatrix} = D'$$

There are certain theorems on transposed matrices. They are:

(a) the transpose of a transposed matrix is equal to the original matrix such that

$$(A')' = A. \tag{3.15}$$

(b) the transpose of the sum of matrices is equal to the sum of transposed matrices. That implies

$$(A + B)' = A' + B' \tag{3.16}$$

Similarly $(A + B + C)' = A' + B' + C'$

For example, if $A = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 5 & 2 \end{bmatrix}$ then

$$(A + B)' = \left\{\begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 2 \end{bmatrix}\right\}'$$

$$= \begin{bmatrix} 2+4 & 2+0 \\ 3+5 & 2+2 \end{bmatrix}' = \begin{bmatrix} 6 & 2 \\ 8 & 4 \end{bmatrix}'$$

$$= \begin{bmatrix} 6 & 8 \\ 2 & 4 \end{bmatrix}$$

Similarly,

$$A' + B' = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}' + \begin{bmatrix} 4 & 0 \\ 5 & 2 \end{bmatrix}'$$

$$= \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 3+5 \\ 2+0 & 2+2 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 2 & 4 \end{bmatrix}$$

$\therefore$ $$(A + B)' = A' + B'$$

Similarly $$(A + B + C)' = A' + B' + C'$$

We can also establish the theorem that the transpose of the difference of two matrices is equal to the difference of the transpose of the matrices such that

$$(A - B)' = A' - B' \tag{3.17}$$

(c) the transpose of the product of two matrices is equal to the product of the transposes of the matrices in reverse order. This implies

$$(AB)' = B' A' \tag{3.18}$$

For example if $$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \text{ then}$$

$$(AB)' = \left\{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\right\}'$$

$$= \begin{bmatrix} 6+1 & 4+4 \\ 0+3 & 0+12 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 3 & 12 \end{bmatrix}'$$

$$= \begin{bmatrix} 7 & 3 \\ 8 & 12 \end{bmatrix}$$

But $$B'A' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}' \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}'$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6+1 & 0+3 \\ 4+4 & 0+12 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 8 & 12 \end{bmatrix}$$

Here $$(AB)' = B' A'.$$

Similarly $$(ABC)' = C' B' A'.$$

3.5 INVERSE OF A MATRIX AND PROPERTIES OF INVERSE

In ordinary algebra, any number has a reciprocal. Thus for any number x, x^{-1} represents its reciprocal or its inverse such that

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

Similarly in matrix algebra, the inverse of a matrix A, denoted by A^{-1} is defined if A is a square matrix and that the inverse of A exists satisfying the condition