3.2 MATRIX OPERATIONS

(I) Equality of matrices

Two matrices A and B are said to be equal if and only if the dimension of both the matrices are the same and each element in corresponding locations of A and B has the same value. Alternatively, $A = B$ if $a_{ij} = b_{ij}$ for all values of i and j.

For example

$$\begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}$$

but

$$\begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} \neq \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$$

If we have the vectors

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$ it means that $x = 10$ and $y = 20$.

(II) Addition of matrices

Two matrices can be added if and only if they have the same dimension. The addition of two matrices A and B will give a third matrix C whose elements are the algebric sum of the corresponding elements of A and B. For example,

if $$A = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}$$

then $$A + B = C$$

$$= \begin{bmatrix} 5+1 & 2+4 \\ -1+2 & 3+2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 \\ 1 & 5 \end{bmatrix}$$

Similarly, if we define

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

then $$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = C$$

In general term

$$[a_{ij}] + [b_{ij}] = [c_{ij}] \text{ where } c_{ij} = a_{ij} + b_{ij}$$

The matrix (or vector) addition satisfies the following properties

$$A + B = B + A \tag{3.11}$$

$$(A + B) + C = A + (B + C). \tag{3.12}$$

(III) Subtraction of matrices

The substraction of two matrices can also be defined in a similar fashion. The subtraction of two matrices A and B is possible if and only if the dimension of A and B are equal. The subtraction of B from A (say) will give another matrix C whose elements will be the algebric difference between the corresponding elements of A and B.

Thus if

$$A = \begin{bmatrix} 7 & 3 \\ 2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix},$$

then

$$A - B = \begin{bmatrix} 7-4 & 3-2 \\ 2-3 & 9-5 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix} = C$$

Thus in general term

$$[a_{ij}] - [b_{ij}] = [c_{ij}] \text{ where } c_{ij} = a_{ij} - b_{ij}.$$

(IV) Scalar multiplication

When a matrix is multiplied by a number (which is termed as scalar in matrix algebra), each and every element of the matrix is multiplied by that number. Such a multiplications is called "scalar multiplication". For example if a matrix $A = \begin{bmatrix} 5 & 2 \\ 3 & 10 \end{bmatrix}$ is multiplied by a scalar 3, then the resultant scalar matrix will be

$$3A = 3\begin{bmatrix} 5 & 2 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 5\times3 & 2\times3 \\ 3\times3 & 10\times3 \end{bmatrix} = \begin{bmatrix} 15 & 6 \\ 9 & 30 \end{bmatrix}$$

In general term, the scalar multiplication is defined as

$$\lambda[a_{ij}] = [\lambda a_{ij}] = [a_{ij}]\lambda \tag{3.13}$$

where λ is a scalar.

The scalar multiplication also holds true for vectors.

(V) Matrix multiplication

Before we explain the technique of multiplicatioin of two matrices, say A and B, it would be most appropriate to state the condition necessary for matrix multiplication, that is the condition of conformability. The matrices A and B are conformable for multiplication in the form AB if the number of columns of the first matrix (A) is equal to the number of rows of the second matrix (B). So that conformability condition can be expressed in terms of the dimension of the matrices A and B. If A is of dimension $m \times n$ and B is of dimension $n \times p$, then AB can be defined. But we cannot define BA as matrix product since the number of column of B is p and the number of row of A is m. So in this particular case we can multiply A by B, that is AB, but we cannot multiply B by A, that is BA. In matrix multiplication the first matrix is called 'lead' matrix and the second one is called 'lag' matrix.

From the above condition of conformability it appears that we can have the products

$$\underset{(3\times 2)}{A}\ \underset{(2\times 2)}{B} \quad \text{and} \quad \underset{(6\times 5)}{A}\ \underset{(5\times 3)}{B}$$

but we cannot have the product

$$\underset{(2\times 2)}{B}\ \underset{(3\times 2)}{A} \quad \text{and} \quad \underset{(5\times 3)}{B}\ \underset{(6\times 5)}{A}$$

Once AB are conformable for multiplication, then what will be the dimension of the resultant matrix (C)? The dimension of the resultant matrix (C) depends on the dimensions of A and B. If $AB = C$, the dimension of C will be equal to the number of rows of lead matrix (A) and number of columns of lag matrix (B). So if the dimensions of A is $(m \times n)$ and the dimensioin of B is $n \times p$, then the dimension of the resultant matrix C will be $m \times p$, such that

$$\underset{(m\times n)}{A}\ \underset{(n\times p)}{B} = \underset{(m\times p)}{C}$$

Thus, in our examples above

$$\underset{(3\times 2)}{A}\ \underset{(2\times 2)}{B} = \underset{(3\times 2)}{C}$$

$$\underset{(6\times 5)}{A}\ \underset{(5\times 3)}{B} = \underset{(6\times 3)}{C}$$

But if both matrices are square matrices and the dimension of both the matrices are the same, then the matrices are conformable for multiplication in both the ways. In other words if the dimension of A is $(m \times m)$ and that of B is also $(m \times m)$ then we can define AB as well as BA and the dimension of the resultant matrix will be $(m \times m)$.

The above conformability condition explains the matrix equation (3.10) above

$$\underset{(n\times m)}{A}\ \underset{(m\times 1)}{X} = \underset{(n\times 1)}{C}$$

Once two matrices are conformable for multiplication, we are to define the procedure for matrix multiplication. Let us assume that $AB = C$ so that

$$\underset{(2\times 2)}{A}\ \underset{(2\times 2)}{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \underset{(2\times 2)}{C}$$

The elements of C are defined as

$$[C_{ij}] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$