03/05/21

Note-4

Gauss's Theorem or Divergence Theorem:

(Relation between surface integral and volume integral)

Statement: The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface 'S'.

Mathematically,

$$\iint_S F.\hat{n}\, ds = \iiint_V \operatorname{div} \bar{F}\, dv. \longrightarrow (1)$$

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Proof:-

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Putting the value of F, n in the statement of the divergence theorem, we have

$$\iint_S (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\hat{n}\, ds$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,dx\,dy\,dz$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz \longrightarrow (2)$$

We require to prove (1).

Let us first evaluate $\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right]dx\,dy$$

$$= \iint_R \Big[F_3(x,y,z)\Big]_{z=f_1(x,y)}^{z=f_2(x,y)}\,dx\,dy.$$

$$= \iint_R \left[F_3(x,y,f_2) - F_3(x,y,f_1)\right]dx\,dy \longrightarrow (3)$$

For the upper part of the surface i.e. S_2, we have

$$dx\,dy = ds_2 \cos\gamma_2 = n_2 \cdot k\, ds_2 .$$

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Again for the lower part of the surface i.e. S_1 we have,

$$dxdy = -\cos\gamma_1 ds_1 = \hat{n}_1 \cdot k \, ds_1$$

$$\iint_R F_3(x, y, f_2)\,dxdy = \iint_{S_2} F_3 \, \hat{n}_2 \cdot k \, ds_2$$

and $$\iint_R F_3(x, y, f_1)\,dxdy = -\iint_{S_1} F_3 \, \hat{n}_1 \cdot k \, ds_1$$

Putting these values in (2), we have.

$$\iiint \frac{\partial F_3}{\partial z} dv = \iint_{S_2} F_3 \hat{n}_2 \cdot k \, ds_2 + \iint_{S_1} F_3 \hat{n}_1 \cdot k \cdot ds_1$$

$$= \iint_S F_3 \hat{n} \cdot k \; ds \longrightarrow (4)$$

Similarly, it can be shown that.

$$\iiint_V \frac{\partial F_2}{\partial y} dv = \iint_S F_2 \hat{n} \cdot \hat{j} \, ds \longrightarrow (5)$$

$$\iiint_V \frac{\partial F_1}{\partial x} dv = \iint_S F_1 \hat{n} \cdot \hat{i} \, ds \longrightarrow (6)$$

eq^n (4) + (5) + (6)

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dv = \iint_S (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \hat{n} \cdot ds$$

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or, $$\boxed{\iiint_V (\nabla \cdot \vec{F})\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds}$$

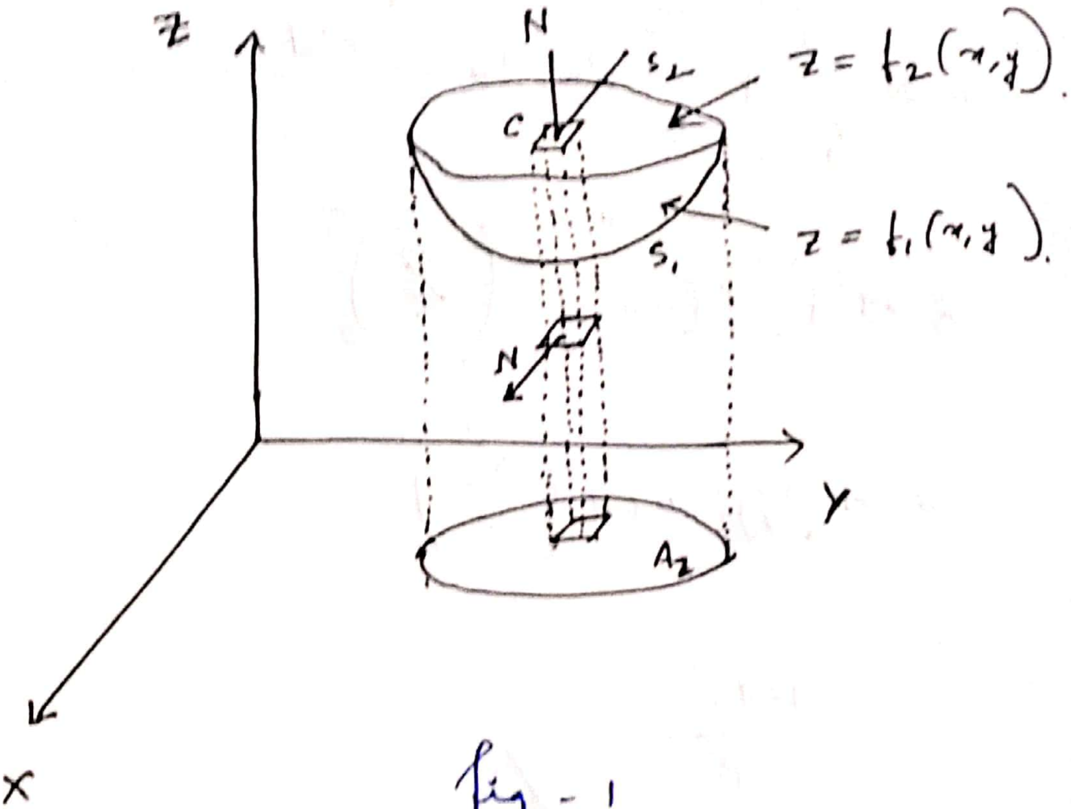

fig - 1

Example:

Use Divergence theorem to evaluate $\int \vec{A} \cdot \vec{ds}$, where $A = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and 'S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$

Solution: $$\iint_S \vec{A} \cdot \vec{ds} = \iiint \operatorname{div} A\, dv$$

$$= \iiint \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right) dv$$

$$= \iiint (3x^2 + 3y^2 + 3z^2)\, dv = 3\iiint (x^2 + y^2 + z^2)\, dv.$$

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On putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we get

$$= 3\iiint r^2 (r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

$$= 3\times 8 \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta \, d\theta \int_0^a r^4 \, dr$$

$$= 24 (\phi)_0^{\pi/2} (-\cos\theta)_0^{\pi/2} \left(\frac{r^5}{5}\right)_0^a$$

$$= 24 \left(\frac{\pi}{2}\right)(-0+1)\left(\frac{a^5}{5}\right)$$

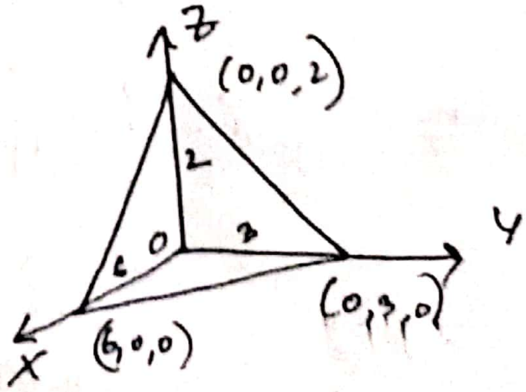
$$= \frac{12\pi a^5}{5}$$

Example: Use the Divergence theorem to evaluate

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

where 'S' is the portion of the plane $x+2y+3z=6$ which lies in the first Octant.

Solution:



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Solution:

$$\iint_S (f_1\,dy\,dz + f_2\,dx\,dz + f_3\,dx\,dy) = \iiint_V \left[\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right] dx\,dy\,dz.$$

where 'S' is a closed surface bounding a volume V.

$$\therefore \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = \iiint_V \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right] dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\,dx\,dy\,dz.$$

$$= 3\iiint_V dx\,dy\,dz$$

$$= 3(\text{Volume of tetrahedron } OABC)$$

i.e $3 \times \left[\frac{1}{3}\text{ Area of base } \Delta OAB \times \text{height } OC\right]$

$$= 3 \times \left[\frac{1}{3} \times \left(\frac{1}{2} \times 6 \times 3\right) \times 2\right]$$

$$= 18$$

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