Important theorems based on equations –

(a) curl (grad ϕ) $= \vec{\nabla} \times \vec{\nabla}\phi = 0$

(b) div (curl $\vec{F}$) $= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0.$ $\Big\} \longrightarrow (1)$

Theorem-1 : If the curl $\vec{A}$ is zero, then $\vec{A}$ is always the gradient of some scalar field ϕ

i.e If $\vec{\nabla} \times \vec{A} = 0$, then $A = \vec{\nabla}\phi \longrightarrow (2)$

Theorem-2: If we come across a vector field $\vec{D}$ for which div. $\vec{D}$ is zero, we can conclude that $\vec{D}$ is the curl of some vector field $\vec{C}$, therefore, we can write,

If $\vec{\nabla} \cdot \vec{D} = 0$, then there is $\vec{C}$ such that

$\vec{D} = \vec{\nabla} \times \vec{C}$. $\longrightarrow (3)$

Vector Integration → 3 types – Line Integral
Surface Integral
Volume Integral.

Line Integral :

Let $\vec{F}(x, y, z)$ be a vector function $\vec{F}$ along the curve AB.

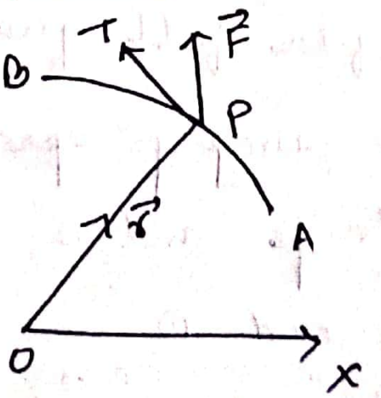


fig (1)

Then, line Integral of a vector function $\vec{F}$ along the curve AB is defined as the integral of the component of $\vec{F}$ along the tangent to the curve AB.

Component of $\vec{F}$ along a tangent PT at P is

= Dot Product of $\vec{F}$ and unit vector along PT

$$= \vec{F} \cdot \frac{d\vec{r}}{ds} \quad \left(\frac{d\vec{r}}{ds} \text{ is a unit vector along PT}\right)$$

Line integral $= \sum \vec{F} \cdot \frac{d\vec{r}}{ds}$ from A to B along AB

$$= \int_c \left(\vec{F} \cdot \frac{dr}{ds}\right) ds$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

<u>Physical Integration</u>:

Since, a vector field $\vec{F}$ can be regarded as a gradient of some scalar function ϕ, i.e., $\vec{F} = \text{grad}\,\phi$, and gradient represents the rate of change of a field quantity. Therefore, line-integral is simply representing the sum of this rate of change and is equal to the total change between A and B.

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \vec{\nabla}\phi \cdot d\vec{r} = \int_A^B \frac{\delta\phi}{\delta r} \cdot d\vec{r} = \int_A^B d\phi$$

$$= \phi(B) - \phi(A)$$

where $\phi(A)$ and $\phi(B)$ are the scalar fields at A and B respectively.

Thus $$\int_A^B F.d\gamma = \int_A^B \vec{\nabla}\phi . d\vec{\gamma} = \phi(B) - \phi(A)$$

- Thus, the value of line integral depends only on the position of the two points in the vector field. (such fields are also called conservative field).

Imp Note.

① Work: If $\vec{F}$ represents the variable force acting on a particle along arc AB, then total work done $= \int_A^B \vec{F}.d\vec{\gamma}$

② Circulation: If $\vec{F}$ represents the velocity of a liquid then $\oint_c \vec{F}.d\vec{\gamma}$ is called the circulation of $\vec{F}$ along the curve c.

If $\oint \vec{F}.d\vec{\gamma} = 0$ then $\vec{F}$ is called conservative

③ When path of integration is a closed curve, then we use '$\oint$' instead of '$\int$'.

Surface Integral

Let $\vec{F}$ be a Vector function and S be the given surface.

Then Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface; i.e.

$\vec{F} \cdot n$ where n is the unit normal vector to an element ds and

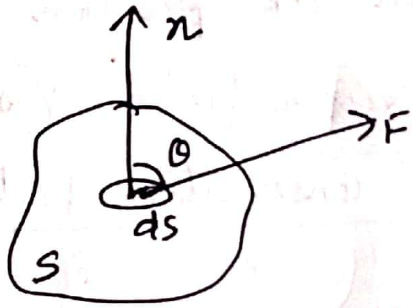


fig: 2

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|}$$

$$ds = \frac{dx\,dy}{(\hat{n} \cdot k)}$$

Surface integral of F over S = $\sum \vec{F} \cdot \hat{n} = \iint_S (\vec{F} \cdot \hat{n})\, ds$

Note: ① Flux = $\iint_S (\vec{F} \cdot \hat{n})\, ds$ where F represents velocity of a fluid.

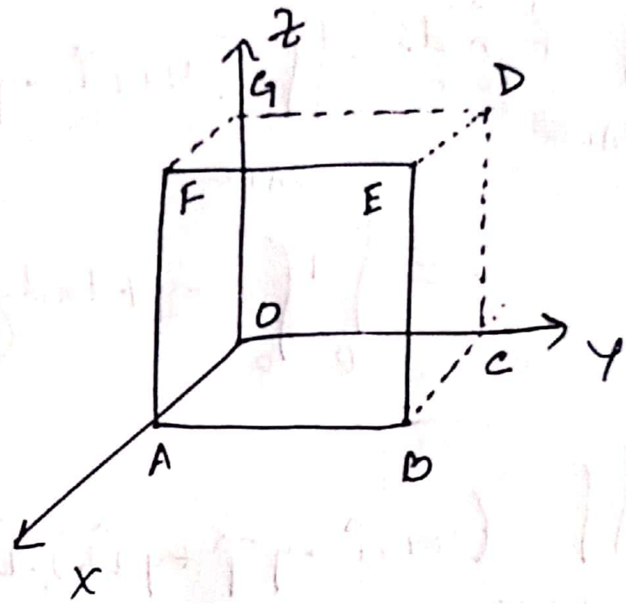
If $\iint_S (\vec{F} \cdot \hat{n})\, ds = 0$, then F is called solenoidal vector point function.

Example:

Show that $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$ where

$\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the planes. $x=0, x=1,$ $y=0, y=1, z=0, z=1.$

Solution:



S. No.	Surface	Outward normal	ds	Eq of Surface
1	OABC	$-k$	$dx\,dy$	$z=0.$
2	DEFG	k	$dx\,dy$	$z=1$
3	OAFG	$-j$	$dx\,dz$	$y=0$
4	BCDE	j	$dx\,dz$	$y=1$
5	ABEF	i	$dy\,dz$	$x=1$
6	OCDG	$-i$	$dy\,dz$	$x=0$

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iint_{OABC} \bar{F}\cdot\hat{n}\,ds + \iint_{DEFG} \bar{F}\cdot\hat{n}\,ds + \iint_{OAFG} \bar{F}\cdot\hat{n}\,ds +$$

$$\iint_{BCDE} \bar{F}\cdot\hat{n}\,ds + \iint_{ABEF} \bar{F}\cdot\hat{n}\,ds + \iint_{OCDG} \bar{F}\cdot\hat{n}\,ds \longrightarrow (a)$$

$$\iint_{OABC} \bar{F}\cdot\hat{n}\,ds = \iint_{OABC} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{k})\,dx\,dy$$

$$= \int_0^1\int_0^1 -yz\,dx\,dy = 0 \quad (\because z=0)$$

$$\iint_{DEFG} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{k}\,dx\,dy = \iint_{DEFG} yz\,dx\,dy$$

$$= \int_0^1\int_0^1 y(1)\,dx\,dy$$

$$= \int_0^1 dx \left[\frac{y^2}{2}\right]_0^1$$

$$= [x]_0^1 \tfrac{1}{2} = \frac{1}{2}$$

$$\iint_{OAFG} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{j})\,dx\,dz = \iint_{OAFG} y^2\,dx\,dz = 0$$

$$(\because y=0)$$

$$\iint_{BCDE} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{j}\,dx\,dz = \iint_{BCDE} (-y^2)\,dx\,dz$$

$$= -\int_0^1 dx \int_0^1 dz = -(x)_0^1 \, (z)_0^1 = -1 \quad (\because y = 1)$$

$$\iint_{ABEF} (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}) \cdot \hat{i}\, dy\,dz = \iint 4xz\, dy\, dz$$

$$= \int_0^1 \int_0^1 4(1)\, z\, dy\, dz$$

$$\cancel{= 4(1)\,(1/2)}$$

$$= 4\,(y)_0^1 \left(\frac{z^2}{2}\right)_0^1$$

$$= 4(1)\,\tfrac{1}{2} = 2$$

$$\iint_{OCDG} (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})(-\hat{i})\, dy\,dz = \int_0^1 \int_0^1 4xz\, dy\, dz$$

$$= 0 \quad (\because x = 0$$

$\therefore$ We get.

$$\iint_S F \cdot \hat{n}\, ds = 0 + \tfrac{1}{2} + 0 - 1 + 2 + 0 = 3/2$$

Hence Proved.

Volume Integral:

Let $\bar{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_V \bar{F}\,dv$.

Example:

If $\bar{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \bar{F}\,dv$ where V is the region bounded by the surfaces.

$x=0,\ y=0,\ x=2,\ y=4,\ z=x^2,\ z=2.$

Solution:

$$\iiint \bar{F}\,dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz.$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})\,dz.$$

$$= \int_0^2 dx \int_0^4 dy \left[z^2\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^{2}$$

$$= \int_0^2 dx \int_0^4 dy \left[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k}\right]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right]_0^4$$

$$= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\,dx.$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}$$

$$= \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$

IMPORTANT THEOREMS

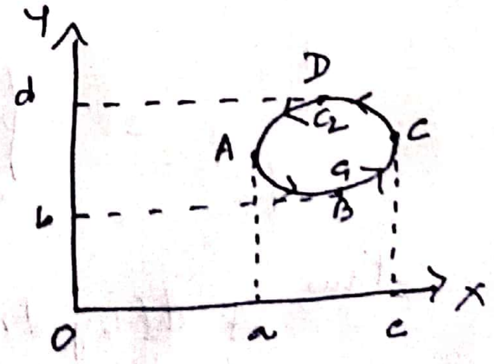
① Green's Theorem: (For a Plane)

Statement: If $\phi(x,y)$, $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in x-y plane, then

$$\boxed{\oint (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy}$$

Proof:

Let the curve C be divided into two curves C_1 (ABC) and C_2 (CDA).



Let the equation of the curve C_1 (ABC) be $y = y_1(x)$

and the equation of the curve C_2 (CDA) be $y = y_2(x)$.

Now,

$$\iint_R \frac{\partial \phi}{\partial y}\, dx\,dy = \int_{x=a}^{x=c} \left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial \phi}{\partial y}\, dy \right] dx$$

$$= \int_a^c \Big[\phi(x,y) \Big]_{y=y_1(x)}^{y=y_2(x)} dx$$

$$= \int_a^c \left[\phi(x,y_2) - \phi(x,y_1) \right] dx$$

$$= -\int_c^a \phi(x,y_2)\,dx - \int_a^c \phi(x,y_1)\,dx$$

$$= -\left[\int_{C_2} \phi(x,y)\,dx + \int_{C_1} \phi(x,y)\,dx \right]$$

$$= -\oint_C \phi(x,y)\,dx$$

Thus, $\oint_C \phi(x,y)\,dx = -\iint_R \frac{\partial \phi}{\partial y}\, dx\,dy \longrightarrow$ (a)

Similarly $\oint_C \psi(x,y)\,dy = \iint \frac{\partial \psi}{\partial x}\, dx\,dy \longrightarrow$ (b)

(a) + (b), gives

$$\oint (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\,dy$$

* In Vector form :

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \iint_R (\nabla \times F)\cdot k\, dR.}$$

where, $\bar{F} = \phi\hat{i} + \psi\hat{j}$, $\bar{r} = x\hat{i} + y\hat{j}$, $\hat{k}$ is a unit vector along z axis and $dR = dx\,dy$.

Example :

A vector field $\bar{F}$ is given by $\bar{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ where C is the circular path given by $x^2+y^2=a^2$.

Solution :

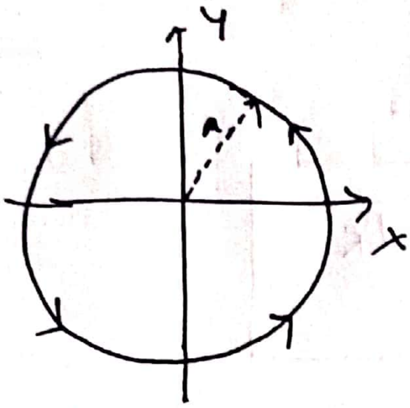
$$\bar{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}.$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C [\sin y\,\hat{i} + x(1+\cos y)\hat{j}]\cdot(\hat{i}\,dx + \hat{j}\,dy)$$

$$= \int_C \sin y\,dx + x(1+\cos y)\,dy.$$

On applying Green's theorem,

$$\oint_c (\phi\, dx + \psi\, dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$



$$= \iint_S \left[(1+\cos y) - \cos y\right] dx\, dy.$$

where 's' is the circular plane surface of radius a.

$$= \iint_S dx\, dy$$

= Area of circle

$$= \pi a^2.$$

Stoke's theorem: (Relation between line integral and Surface Integral).

Statement: Surface integral of the component of curl $\bar{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\bar{F}$ taken along the closed curve C.

Mathematically,

$$\boxed{\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds}$$

where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal to any surface ds.

Proof:

Let, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$.

$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

On putting the values of $\bar{F}\cdot d\bar{r}$ in the statement of the theorem

$$\oint_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

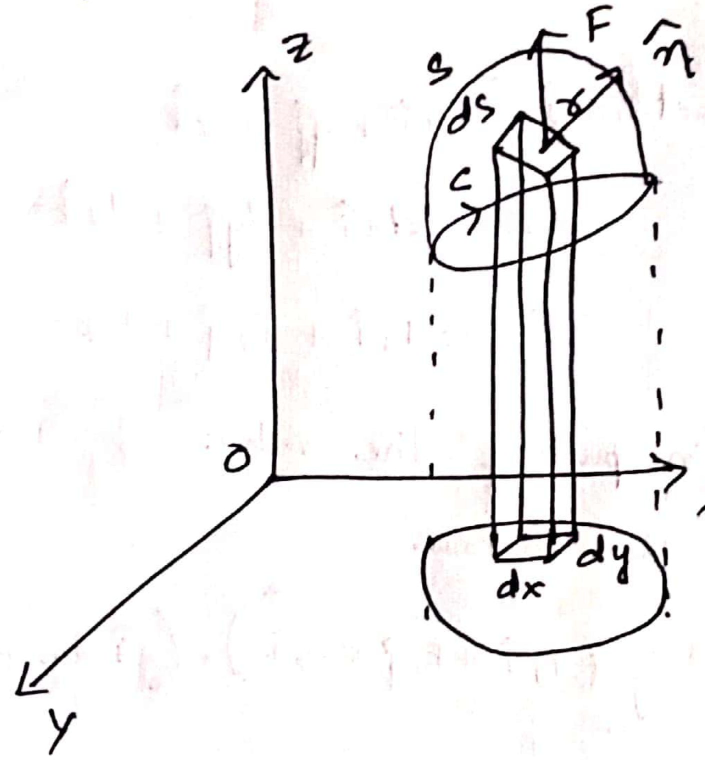
$$= \iint_S \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})\,ds.$$

$$\Rightarrow \oint (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right]\cdot(i\cos\alpha + \hat{j}\cos\beta + k\cos\gamma)\,ds.$$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cos\gamma \right] ds \longrightarrow (1)$$

Let us first Prove:

$$\oint F_1\, dx = \iint_S \left(\frac{\partial F_1}{\partial z} \cos\beta - \frac{\partial F_1}{\partial y} \cos\gamma \right) ds \longrightarrow (2)$$



Let the equation of the surface S be $z = g(x, y)$.

$$\oint_C F_1(x, y, z)\, dx = \oint_C F_1[x, y, g(x, y)]\, dx$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x, y, g(x, y))\, dx\, dy$$

(By Green's theorem)

Region R is the projection of the surface on x-y plane.

$$\Rightarrow \oint_c F_1(x,y,z)\,dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right) dx\,dy. \quad (3)$$

The direction cosines of the normal to the surface $z = g(x,y)$ are given by –

$$\frac{\cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{\cos\beta}{-\frac{\partial g}{\partial y}} = \frac{\cos\gamma}{1}.$$

$dx\,dy$ = Projection of ds on xy plane $= ds\cos\gamma$.

Putting the values of ds in RHS of eq^n (2).

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) ds = \iint_R \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\frac{dx\,dy}{\cos\gamma}$$

$$= \iint_R \left(\frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_R \left[\frac{\partial F_1}{\partial z}\left(-\frac{\partial g}{\partial y}\right) - \frac{\partial F_1}{\partial y}\right] dx\,dy$$

$$= -\iint_R \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right] dx\,dy \quad (4)$$

From eqⁿ (3) & (4), we get

$$\oint_C F_1\,dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)ds \longrightarrow (5)$$

Similarly, $$\oint_C F_2\,dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right)ds \longrightarrow (6)$$

$$\oint_C F_3\,dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right)ds \longrightarrow (7)$$

(5) + (6) + (7) $\Rightarrow$

$$\oint_C F_1\,dx + \oint_C F_2\,dy + \oint F_3\,dz$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma + \frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha + \frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right)ds.$$

Proved

Happy's

Example.

Using Stoke's theorem, ~~calc~~ evaluate

$$\int_C [(2x-y)\,dx - yz^2\,dy - y^2z\,dz]$$

where C is the circle $x^2+y^2=1$, corresponding to the surface of sphere of unit radius.

Solution :

$$\int_C [(2x-y)\,dx - yz^2\,dy - y^2z\,dz]$$

$$= \int_C [(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}].(i\,dx + j\,dy + k\,dz)$$

By Stoke's theorem,

$$\oint \bar{F}.d\bar{r} = \iint_S \text{curl}\,\bar{F}.\bar{n}\,ds \longrightarrow (1)$$

$$\text{curl}\,\bar{F} = \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz+2yz)\hat{i} - (0-0)\hat{j} + (0+1)\hat{k}$$

$$= \hat{k}.$$

from (1) we get

$$\iint \hat{k}\cdot\hat{\eta}\, ds$$

$$= \iint \hat{k}\cdot\hat{\eta}\, \frac{dx\,dy}{\eta\cdot\hat{k}} \qquad \left[\because ds = \frac{dx\,dy}{(\hat{\eta}\cdot k)}\right.$$

$$= \iint dx\,dy$$

$$= \pi$$

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