Partial Derivatives

First-Order Partial Derivatives

Given a multivariable function, we can treat all of the variables except one as a constant and then differentiate with respect to that one variable. This is known as a *partial derivative of the function*

For a function of two variables $z = f(x, y)$, the partial derivative with respect to x is written:

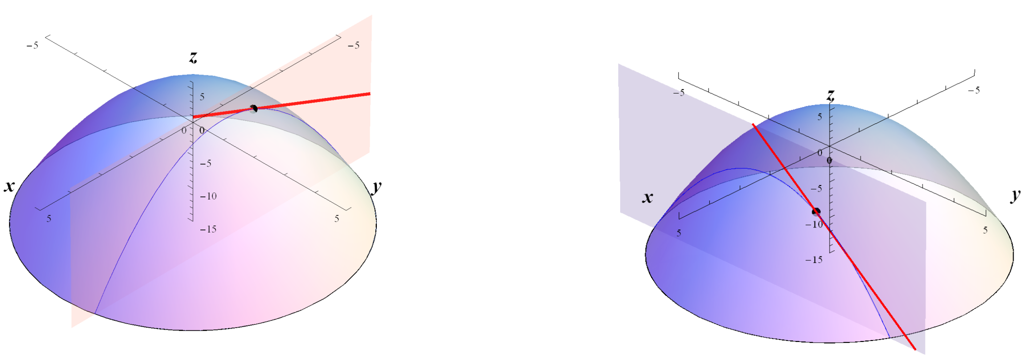
$$f_x,\ f_x(x,y),\ \frac{\partial f}{\partial x},\ \frac{\partial}{\partial x}\left(f(x,y)\right),\ \frac{\partial z}{\partial x},\ \text{or } D_x f$$

The partial derivative with respect to y is written:

$$f_y,\ f_y(x,y),\ \frac{\partial f}{\partial y},\ \frac{\partial}{\partial y}\left(f(x,y)\right),\ \frac{\partial z}{\partial y},\ \text{or } D_y f$$

(The notation for functions of more than two variables is similar.)

Graphically, $\frac{\partial f}{\partial x}$ tells us the instantaneous rate of change of the function if we hold y fixed and move parallel to the x-axis in the positive direction, while $\frac{\partial f}{\partial y}$ tells us the instantaneous rate of change of the function if we hold x fixed and move parallel to the y-axis in the positive direction. See Figure 1 below.



(a) As we move in the $+x$-direction from $(-1, 1)$, $\frac{\partial f}{\partial x}(-1, 1)$ is positive, so f is increasing in that direction.

(b) As we move in the $+y$-direction from $\left(2, \frac{3}{2}\right)$, $\frac{\partial f}{\partial x}\left(2, \frac{3}{2}\right)$ is negative, so f is decreasing in that direction.

Figure 1

Limit Definition

As with derivatives in calculus I, there is a limit definition for partial derivatives:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y}{h}$$

We won't be using the limit definition to find partial derivatives in this class, but we would need it if we wanted to go through a later example.

Example 1. Find all first partial derivatives of the following functions:

1. $f(x,y)=x^2y^2+y^2+2x^3y \quad \frac{\partial f}{\partial x}=2xy^2+6x^2y,\ \frac{\partial f}{\partial y}=2x^2y+2y+2x^3,$
 $\frac{\partial^2 f}{\partial x^2}=12xy+2y^2,\ \frac{\partial^2 f}{\partial y\,\partial x}=6x^2+4xy,\ \frac{\partial^2 f}{\partial y^2}=2+2x^2,\ \frac{\partial^2 f}{\partial x\,\partial y}=6x^2+4xy,$
 $\frac{\partial^3 f}{\partial x^3}=12y,\ \frac{\partial^3 f}{\partial y\,\partial x^2}=12x+4y=\frac{\partial^3 f}{\partial x^2\,\partial y}=\frac{\partial^3 f}{\partial x\,\partial y\,\partial x},\ \frac{\partial^3 f}{\partial y^3}=0,\ \frac{\partial^3 f}{\partial x\,\partial y^2}=4x=\frac{\partial^3 f}{\partial y^2\,\partial x}=\frac{\partial^3 f}{\partial y\,\partial x\,\partial y}$
2. $f(x,y)=e^{x^2y} \quad \frac{\partial f}{\partial x}=2xye^{x^2y},\ \frac{\partial f}{\partial y}=x^2e^{x^2y},$
 $\frac{\partial^2 f}{\partial x^2}=4x^2y^2e^{x^2y}+2ye^{x^2y},\ \frac{\partial^2 f}{\partial y\,\partial x}=2xe^{x^2y}+2x^3ye^{x^2y},\ \frac{\partial^2 f}{\partial y^2}=x^4e^{x^2y},\ \frac{\partial^2 f}{\partial x\,\partial y}=2xe^{x^2y}+2x^3ye^{x^2y}$
3. $f(x,y)=xe^{x^2y} \quad \frac{\partial f}{\partial x}=e^{x^2y}+2x^2ye^{x^2y},\ \frac{\partial f}{\partial y}=x^3e^{x^2y},$
 $\frac{\partial^2 f}{\partial x^2}=6xye^{x^2y}+4x^3y^2e^{x^2y},\ \frac{\partial^2 f}{\partial y\,\partial x}=3x^2e^{x^2y}+2x^4ye^{x^2y},\ \frac{\partial^2 f}{\partial y^2}=x^5e^{x^2y},\ \frac{\partial^2 f}{\partial x\,\partial y}=3x^2e^{x^2y}+2x^4ye^{x^2y}$
4. $h(x,y,z)=\frac{yze^x}{x^2\sin(y)} \quad \frac{\partial h}{\partial x}=\frac{yze^xx^2\sin(y)-yze^x2x\sin(y)}{(x^2\sin(y))^2}=\ldots=\frac{yze^x(x-1)}{x^3\sin(y)},$
 $\frac{\partial h}{\partial y}=\frac{ze^xx^2\sin(y)-yze^xx^2\cos(y)}{(x^2\sin(y))^2}=\frac{ze^x\,(\sin(y)-y\cos(y))}{x^4\sin^2(y)},\ \frac{\partial h}{\partial z}=\frac{ye^x}{x^2\sin(y)}$

 Some additional examples we'll look at if time permits:

5. $f(x,y,z)=2x^2y+e^yz+\sqrt{z}\ln(x) \quad \frac{\partial f}{\partial x}=4xy+\frac{\sqrt{z}}{x},\ \frac{\partial f}{\partial y}=2x^2+e^yz,\ \frac{\partial f}{\partial z}=e^y+\frac{\ln(x)}{2\sqrt{z}}$
6. $f(x,y,z)=ze^{x^2+xy} \quad \frac{\partial f}{\partial x}=z(2x+y)e^{x^2+xy},\ \frac{\partial f}{\partial y}=xze^{x^2+xy},\ \frac{\partial f}{\partial z}=e^{x^2+xy}$
7. $f(x,y,z)=\frac{x}{(xy-z)^2} \quad \frac{\partial f}{\partial x}=-\frac{xy+z}{(xy-z)^3},\ \frac{\partial f}{\partial y}=-\frac{2x^2}{(xy-z)^3},\ \frac{\partial f}{\partial z}=\frac{2x}{(xy-z)^3}$

Higher Order Derivatives

We can find second order derivatives by simply differentiating the first order partial derivatives again. We can find third or higher order derivatives in a similar manner.

The notation for second order derivatives is:

$$
\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} = D_{xx} f \\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} = D_{yy} f \\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x} = D_{xy} f \\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y} = D_{yx} f
\end{aligned}
$$

For third order derivatives, we have

$$
\begin{aligned}
&f_{xxx},\ f_{yyx},\ f_{yxy},\ \text{etc.} \\
&\frac{\partial^3 f}{\partial x^3},\ \frac{\partial^3 f}{\partial x\,\partial y^2},\ \frac{\partial^3 f}{\partial y\,\partial x\,\partial y},\ \text{etc.}
\end{aligned}
$$

There are $2^3 = 8$ possible third order partial derivatives. In general there are

$$(\text{number of indep variables})^n$$

nth-order partial derivatives.

Note that order in which we read off which variable to differentiate with respect to changes between the subscript notation and the ∂ notation. In the subscript notation we read from left to right, in the ∂ notation we read from right to left. So, for example, f_{yyx} is equivalent to $\frac{\partial^3 f}{\partial y^2\,\partial x}$ (in both, we differentiate with respect to y twice and then with respect to x).

Example 2. Find all of the second order partial derivatives of the functions in Example 1. Find all of the third order partial derivatives for Example 1.1. [Partial solutions on previous page.]

Clairaut's Theorem

Theorem 1 (Clairout's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, the*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

This essentially says that for "nice" functions the mixed partial derivatives are equal, which means that the order in which we differentiate won't matter in most/all of the problems you'll see here.

One example of a function whose mixed partials are different (so, it does *not* satisfy the hypotheses of Clairout's Theorem) is

$$
f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) \neq (0, 0) \end{cases}
$$

One can use the limit definition of the derivatives to show that $f_{xy}(0, 0) = 1$ but $f_{yx}(0, 0) = -1$

Implicit Differentiation

We can do implicit differentiation in a manner very similar to what we would have done back in calculus I.

For example, if z is defined implicitly as a function of x and y by the equation

$$z^2 - x^2y + z^3x + zy = 3$$

and we differentiate implicitly with respect to x, then

$$\begin{aligned} 2z\frac{\partial z}{\partial x} - 2xy + 3z^2\frac{\partial z}{\partial x}x + z^3 + \frac{\partial z}{\partial x}y &= 0 \\ \frac{\partial z}{\partial x}\left(2z + 3z^2x + z^3 + y\right) &= 2xy \\ \frac{\partial z}{\partial x} &= \frac{2xy - z^3}{2z + 3z^2x + y} \end{aligned}$$

Example 3. Find each of the following for the problem above:

1. $\frac{\partial z}{\partial y} = \dfrac{x^2 - z}{2z + 3z^2x + y}$

2. $\frac{\partial^2 z}{\partial x\,\partial y}$

$$\begin{aligned} \frac{\partial^2 z}{\partial x\,\partial y} &= \frac{\left(x^2 - \frac{\partial z}{\partial x}\right)\left(2z + 3z^2x + y\right) - \left(x^2 - z\right)\left(2\frac{\partial z}{\partial x} + 6z\frac{\partial z}{\partial x}x + 3z^2\right)}{\left(2z + 3z^2x + y\right)^2} \\ &= \frac{\left(x^2 - \frac{x^2 - z}{2z + 3z^2x + y}\right)\left(2z + 3z^2x + y\right) - \left(x^2 - z\right)\left(2\frac{x^2 - z}{2z + 3z^2x + y} + 6xz\frac{x^2 - z}{2z + 3z^2x + y} + 3z^2\right)}{\left(2z + 3z^2x + y\right)^2} \end{aligned}$$

This would be sufficient for an answer on a test. If you were to continue to simplify (and fully expanded the numerator) you would get:

$$= \frac{-6x^5z + 9x^4z^4 - 2x^4 + 6x^3yz^2 - 9x^3z^4 + 12x^3z^3 + 9x^3z^2 + x^2y^2 - 3x^2yz^2 + 4x^2yz - x^2y - 6x^2z^3 + 4x^2z^2 + 2x^2z + 9xz^5 - 3xz^3 + 3yz^3 + yz + 6z^4}{(3xz^2 + y + 2z)^3}$$