Vector Differentiation and Integration:

Differentiation of a vector with respect to time:

If a vector changes in magnitude as well as direction continuously with respect to some scalar variable, then such a vector is a function of this scalar variable, so that it can be differentiated with respect to that scalar variable and a new vector is obtained. If the scalar variable is time, the result of differentiation of the vector with time is called time derivative of the vector

Velocity and acceleration

Let $\vec{r}$ be the position vector of a particle at time t, $\vec{r}$ is a function of variable 't'.

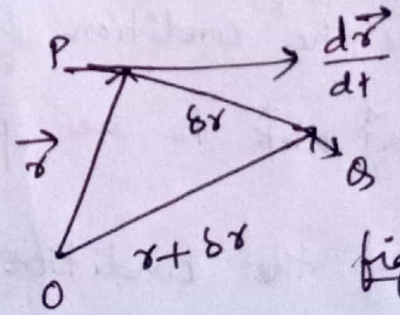

fig: 1

As time increases, the particle moves and the position vector changes in direction and magnitude, when time changes from 't' to 't+δt', $\vec{r}$ becomes $\vec{r}+\vec{\delta r}$ at Q as shown in figure 1.

The average rate of change of the position vector with time 't' is given by,

$$\frac{r+\delta r-\vec{r}}{t+\delta t-t} = \frac{\delta \vec{r}}{\delta t} \longrightarrow (1)$$

In the limit when $\delta t \rightarrow 0$, the rate of change of $\vec{r}$ with time is called first time derivative and written as $\frac{d\vec{r}}{dt}$, i.e $\lim_{\delta t \rightarrow 0} \frac{\delta \vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$

Again, the rate of change of position w.r.t. time is velocity.

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} \longrightarrow (2)$$

We have in cartesian co-ordinates,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \longrightarrow (3)$$

Similarly, the second time derivative of position w.r.t time gives the acceleration, $\vec{a}$ of the particle. Thus

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} \longrightarrow (4)$$

$$\Rightarrow \vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}.$$

$$\Rightarrow \vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}.$$

$$\Rightarrow \vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}. \longrightarrow (5)$$

Force $\vec{F}$ acting on the particle of mass m is given by Newton's second law of motion.

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} \longrightarrow (6)$$

Differentiation of Sums and Products

(1) Differentiation of sums of the vectors:

Let, $\vec{R} = \vec{A} + \vec{B}$; $\vec{A}$ and $\vec{B}$ are the functions of 't'

For a change of time from t to $t+\delta t$, we have.

$$\vec{R} + \delta\vec{R} = (\vec{A} + \delta\vec{A}) + (\vec{B} + \delta\vec{B})$$

Hence, $\delta\vec{R} = \delta\vec{A} + \delta\vec{B}$ $(\because \vec{R} = \vec{A} + \vec{B})$

dividing this by δt and putting the limit, we get,

$$\lim_{\delta t\to 0}\frac{\delta\vec{R}}{\delta t} = \lim_{\delta t\to 0}\frac{\delta\vec{A}}{\delta t} + \lim_{\delta t\to 0}\frac{\delta\vec{B}}{\delta t}.$$

$$\frac{d\vec{R}}{dt} = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}.$$

$$\frac{d\vec{R}}{dt} = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt} \longrightarrow (7)$$

So, that such differentiation is 'distributive' and holds good for any given number i.e.,

$$\frac{d}{dt}(\vec{A} + \vec{B} + \vec{C} + \ldots) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt} + \frac{d\vec{C}}{dt} + \ldots \longrightarrow (8)$$

Similarly, we can show that for difference of vectors

$$\frac{d}{dt}(\vec{A} - \vec{B}) = \frac{d\vec{A}}{dt} - \frac{d\vec{B}}{dt} \longrightarrow (9)$$

(2) Differentiation of Scalar Product of two Vectors, $\vec{A} \cdot \vec{B}$.

Let $\vec{R} = \vec{A} \cdot \vec{B}$, the increase δt in 't' gives,

$$\vec{R} + \delta\vec{R} = (\vec{A} + \delta\vec{R}) \cdot (\vec{B} + \delta\vec{B})$$

$$\vec{R} + \delta\vec{R} = \vec{A} \cdot \vec{B} + \vec{A} \cdot \delta\vec{B} + \vec{B} \cdot \delta\vec{A} + \delta\vec{A} \cdot \delta\vec{B}$$

Now, on neglecting the $\delta\vec{A} \cdot \delta\vec{B}$ which is small, we get,

$$\delta\vec{R} = \vec{A} \cdot \delta\vec{B} + \vec{B} \cdot \delta\vec{A} \qquad (\because \vec{R} = \vec{A} \cdot \vec{B})$$

Dividing by δt and proceeding the limit, we get,

$$\frac{d\vec{R}}{dt} = \frac{d}{dt}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{dt} + \vec{B} \cdot \frac{d\vec{A}}{dt} \longrightarrow (10)$$

(3) Differentiation of Vector Product $\vec{A}\times\vec{B}$.

Let $\vec{R} = \vec{A}\times\vec{B}$.

Now, $\vec{R}+\delta\vec{R} = (\vec{A}+\delta\vec{A})\times(\vec{B}+\delta\vec{B})$

$\Rightarrow \vec{R}+\delta\vec{R} = \vec{A}\times\vec{B} + A\times\delta\vec{B} + \delta\vec{A}\times\vec{B} + \delta\vec{A}\times\delta\vec{B}$.

Neglecting $\delta\vec{A}\times\delta\vec{B}$ and dividing by δt as before, we get

$$\frac{d}{dt}(\vec{R}) = \frac{d}{dt}\left(\vec{A}\times d\vec{B} + d\vec{A}\times\vec{B}\right)$$

$$\Rightarrow \frac{d\vec{R}}{dt} = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B} \longrightarrow (11)$$

(4) Differentiation of Triple Products.

(a) In Scalar Triple Product:

$$\frac{d}{dt}\left[\vec{A}\cdot(\vec{B}\times\vec{C})\right] = \frac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C}) + \vec{A}\cdot\left[\frac{d\vec{B}}{dt}\times\vec{C}\right] + \vec{A}\cdot\left[\vec{B}\times\frac{d\vec{C}}{dt}\right] \longrightarrow (12)$$

(b) In Vector Triple Product:

$$\frac{d}{dt}\left[\vec{A}\times(\vec{B}\times\vec{C})\right] = \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C}) + \vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right) \longrightarrow (13)$$

Fields : Scalar and Vector

Field : If a ~~ϕ~~ physical quantity varies from point to point in space it can be expressed as a continuous function of the position of a point in a region of space, then such a function is called the function of position or point function and the region in which it specifies the physical quantity is called the field. Two main kinds of fields are :

(a) Scalar fields :

A Scalar field is represented by a continuous scalar function $\phi(x, y, z)$ giving the value of quantity at each point. In all practical cases, the magnitude of such function does not change abruptly when it passes from any point to another close to it.

eg : distribution of temperature, magnetic and electrostatic potentials etc.

(6) Vector field:

A vector field is represented at every point by a continuous vector function $\vec{F}(x,y,z)$. At any given point of field, the function $\vec{F}(x,y,z)$ is specified by a vector of definite magnitude and direction, both of which change continuously from point to point throughout the field region.

Example: the distribution of velocity in a fluid, distribution of electric and magnetic field intensity etc.

Partial differentiation and Gradient:

Partial Differentiation:

If $\phi(x,y,z)$ is a scalar function of position in space, i.e. of co-ordinates x, y, z, then $\frac{\delta\phi}{\delta x}$ denotes the rate of change of ϕ wrt x when y and z remain constant, is called partial differentiation. Similarly, $\frac{\delta\phi}{\delta y}$ and $\frac{\delta\phi}{\delta z}$ denotes the rate of change of ϕ wrt y and z respectively.

Gradient:

The vector function $\hat{i}\frac{\delta\phi}{\delta x} + \hat{j}\frac{\delta\phi}{\delta y} + \hat{k}\frac{\delta\phi}{\delta z}$ is called gradient of the scalar function.

$$\text{grad}\,\phi = \hat{i}\frac{\delta\phi}{\delta x} + \hat{j}\frac{\delta\phi}{\delta y} + \hat{k}\frac{\delta\phi}{\delta z} \longrightarrow (14)$$

$$d\phi = \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz \longrightarrow (15)$$

If $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ is the radius vector of the point in space from the origin, then,

$$\vec{dr} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$d\phi = (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)\cdot\left(\hat{i}\frac{\delta\phi}{\delta x} + \hat{j}\frac{\delta\phi}{\delta y} + \hat{k}\frac{\delta\phi}{\delta z}\right)$$

$$\Rightarrow d\phi = \vec{dr}\cdot\text{grad}\,\phi$$

$$\Rightarrow d\phi = \text{grad}\,\phi\cdot\vec{dr} \longrightarrow (16)$$

The operator $(\vec{\nabla})$: (read as del or nabla)

$$\vec{\nabla} = \hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}$$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\phi = \vec{\nabla}\phi$$

$$\therefore d\phi = \vec{\nabla}\phi\cdot\vec{dr} = (\vec{\nabla}\cdot\vec{dr})\phi \longrightarrow (17)$$

Physical Meaning of the Gradient of the Scalar Function:

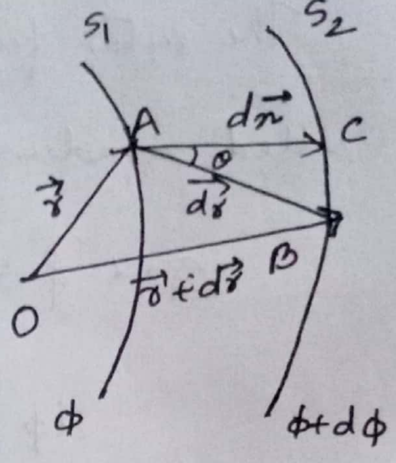

fig: 2

Let us consider two surfaces S_1 and S_2 very close together and ϕ and $\phi + d\phi$ are the corresponding scalar functions. Choose points A and B on the two surfaces as shown in figure 2.

$$\vec{OA} = \vec{r}$$
$$\vec{OB} = \vec{r} + d\vec{r}$$
$$\vec{AB} = d\vec{r}$$

The least distance between two surfaces S_1 and S_2 is $\vec{AC}$ in the direction of normal at $\vec{A}$. Let $\hat{n}$ be the unit vector along $\vec{AC}$ and $\vec{AC} = d\vec{n}$.

$$d\vec{n} = dr \cos\theta = \hat{n} \cdot d\vec{r} \longrightarrow (18)$$

The rate of increase of ϕ at A in the direction of AB will be $\frac{d\phi}{\delta r}$ and this rate of increase becomes greatest only when δr is minimum, i.e., along $\vec{AC}$ which is the least distance between surfaces. So that greatest rate of increment is $\frac{\delta\phi}{\delta n}$, in the direction of $\hat{n}$,

$$\therefore\ d\phi = \frac{\delta\phi}{\delta n}\, dn$$

Substituting from (17) we get,

$$d\phi = \frac{\delta\phi}{\delta n}\,\hat{n}\cdot\vec{dr} \longrightarrow (18)$$

But $d\phi = (\vec{\nabla}\cdot\vec{dr})\phi$.

$$d\phi = (\nabla\phi\cdot\vec{dr}) \longrightarrow (19)$$

Equating these two values, we get

$$\vec{\nabla}\phi\cdot\vec{dr} = \frac{\delta\phi}{\delta n}\,\hat{n}\cdot\vec{dr}$$

$$\nabla\phi = \frac{\delta\phi}{\delta n}\,\hat{n} = \cancel{\nabla\phi}\ \text{grad}\ \phi.$$

Thus, the gradient of scalar field grad ϕ or $\nabla\phi$ is a vector whose magnitude at any point is equal to the maximum rate of increase of ϕ and directed along the direction in which this maximum change occurs, i.e, perpendicular to level surface at that point.

Ex: Let V be the potential due to static charges then electric field intensity at any point is in the direction of the greatest rate of decrease of potential, ~~is so~~ and its magnitude is equal to the rate of decrease. $\boxed{\vec{E} = -\nabla V} \longrightarrow (20)$

* $\nabla\phi \longrightarrow$ Maximum rate of change of ϕ.

Operations using $\vec{\nabla}$

① **Divergence:** (∇. A vector $\longrightarrow$ A scalar)

Let $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ is a vector field.

then, $\vec{\nabla}\cdot\vec{F} = \nabla_x F_x + \nabla_y F_y + \nabla_z F_z \cdot$ $\left(\vec{\nabla} = \nabla_x\hat{i} + \nabla_y\hat{j} + \nabla_z\hat{k}\right)$

or, $$\nabla\cdot F = \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z} \quad \text{(21)}$$

This sum is invariant under a co-ordinate transformation, ie it is quite independent of the unit vectors that may be chosen.

For a different system.

$$\nabla'\cdot\vec{F} = \frac{\delta F_{x'}}{\delta x'} + \frac{\delta F_{y'}}{\delta y'} + \frac{\delta F_{z'}}{\delta z'}$$

$\nabla'\cdot\vec{F} = \vec{\nabla}\cdot\vec{F}$ for every point in space.

$\therefore$ $\vec{\nabla}\cdot\vec{F}$ is a scalar field which must represent some physical quantity.

② <u>Curl</u> :

$$\vec{\nabla} \times \vec{F} = \text{curl } \vec{F} = \text{A vector.}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

Thus, we get,

$$\boxed{\begin{aligned} \vec{\nabla}\phi &= \text{grad } \phi = \text{A vector.} \\ \vec{\nabla}\cdot\vec{F} &= \text{div } \vec{F} = \text{A scalar.} \\ \vec{\nabla}\times\vec{F} &= \text{Curl } \vec{F} = \text{A vector.} \end{aligned}}$$

Second derivative of vector fields :

ⓐ $\nabla\cdot(\nabla\phi)$ — div. of a gradient of ϕ

ⓑ $\nabla\times(\nabla\phi)$ — curl of a gradient of ϕ

ⓒ $\vec{\nabla}(\nabla\cdot\vec{F})$ — gradient of a divergence of $\vec{F}$

ⓓ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})$ — divergence of a curl of $\vec{F}$

ⓔ $\vec{\nabla}\times(\vec{\nabla}\times\vec{F})$ — curl of a curl of $\vec{F}$.

ⓑ $\nabla\times(\nabla\phi) = (\vec{\nabla}\times\vec{\nabla})\phi. = 0$ ($\because$ curl of 2 equal vectors $= 0$)

ⓓ $\nabla\cdot(\nabla\times\vec{F}) = 0$ (if 2 of the 3 vectors are equal, their product is 0)

↓
Important.

The Laplacian Operator

$\nabla.(\nabla\phi) = \nabla.\nabla\phi = (\nabla.\nabla)\phi = \nabla^2\phi.$

∇^2 has a special name $\longrightarrow$ The Laplacian Operator

$(\vec{\nabla}.\vec{\nabla})$

Laplacian $= \nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$. (A Scalar qty) $\longrightarrow$ (22)

Curl of Curl:

We have,

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - (\vec{A}.\vec{B})\vec{C}$$

$$\therefore \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F}) - (\vec{\nabla}.\vec{\nabla})\vec{F}$$

$$\nabla\times(\nabla\times\vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F}) - \vec{\nabla}^2\vec{F}$$ $\longrightarrow$ (23)

Important Conclusion:

1. $\vec{\nabla}.\nabla\phi = \nabla^2\phi =$ Scalar
2. $\vec{\nabla}\times(\nabla\phi) = 0$
3. $\vec{\nabla}(\vec{\nabla}.\vec{F}) =$ A vector
4. $\vec{\nabla}.(\nabla\times\vec{F}) = 0$
5. $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F}) - \nabla^2\vec{F}$
6. $(\vec{\nabla}.\vec{\nabla})\vec{F} = \nabla^2 F =$ A vector.

$\longrightarrow$ (24)

Physical Meaning of Divergence:

If $\vec{v}$ is the velocity of a fluid at a point, then Div $\vec{v}$ at a point inside the fluid is the rate at which the fluid is flowing away from the point per unit volume.

'Div' $\longrightarrow$ Amount of flux.

+ve value of $\vec{\nabla}\cdot\vec{v}$ $\rightarrow$ fluid ie expanding, density is decreasing

-ve value of $\vec{\nabla}\cdot\vec{v}$ $\rightarrow$ fluid is contracting, density is increasing.

eg: Divergence of Current density $\longrightarrow$ (current / unit area)

At a point gives the amount of charge flowing out / sec / unit volume from a closed surface surrounding the point.

$\vec{\nabla}\cdot\vec{v} = 0 \Longrightarrow$ flux entering any element of space = flux leaving it.

✗ If $\vec{\nabla}\cdot\vec{A} = 0$; A is called Solenoidal Vector