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Target Audience:BSc Mathematics(H)C2.1

Real Sequences

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Content of the Course

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1. Basics of Sequences.
2. Examples.
3. Bounded Sequences.
4. Know about the inequality $|x| < k \Leftrightarrow -k < x < k$.
5. Examples.
6. Convergence of Sequences.
7. Examples.

Basics of Sequence

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Definition

A sequence is a function f with domain $\mathbb{N}$ of natural numbers and the range contained in the set of real numbers $\mathbb{R}$ i.e., $f : \mathbb{N} \to \mathbb{R}$.

Let the value of function f at $n \in \mathbb{N}$ is a_n i.e., $f(n) = a_n$. Thus a real sequence is denoted as $< a_n >$ and at $n = 1, 2, \ldots$ we have $f = a_1, a_2, \ldots$ respectively. Here a_n is the n^{th} term or general term of the sequence.

Definition

The set of all distinct terms of sequences $< a_n >$ is called range set of the sequence.

Note: Range set of the sequence may be finite or infinite set but Sequence must be infinite.

Examples

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Example

1. $< a_n >=< n^2 >, \ n \in \mathbb{N}.$
2. $< a_n >=< 2 >, \ n \in \mathbb{N}.$
3. $< a_n >=< (-1)^n >, \ n \in \mathbb{N}.$
4. $< a_n >=< \frac{1}{n} >, \ n \in \mathbb{N}.$

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Examples

Example

1. $< a_n > = < n^2 : n \in \mathbb{N} > = < n^2 > = < 1, 4, 9, ... >$,
   Range of the sequence $= \{1, 4, 9, ...\}$
2. $(a_n) = (2 : n \in \mathbb{N}) = (2, 2, 2, ...)$,
   Range of the sequence $= \{2\}$.
3. $< a_n > = < (-1)^n : n \in \mathbb{N} > = < -1, 1, -1, 1, ... >$,
   Range of the sequence $= \{-1, 1\}$.
4. $< a_n > = < \frac{1}{n} : n \in \mathbb{N} > = < 1, \frac{1}{2}, \frac{1}{3}, ... >$,
   Range of the sequence $= \{1, \frac{1}{2}, \frac{1}{3}, ...\}$.

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Bounded Sequences with Some Examples

Definition

A sequence $X =< a_n >$ of real number is said to be bounded if there exists a real number $M > 0$ such that $|a_n| \leq M$, $\forall n \in \mathbb{N}$.

Definition

A sequence $X =< a_n >$ of real number is said to be bounded above if there exists a real number M such that $a_n \leq M$, $\forall n \in \mathbb{N}$.

Definition

A sequence $X =< a_n >$ of real number is said to be bounded below if there exists a real number m such that $a_n \geq m$, $\forall n \in \mathbb{N}$.

Know about the inequality $|x| < k \Leftrightarrow -k < x < k$

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For real numbers $x, k > 0$ show that $|x| < k \Leftrightarrow -k < x < k$

proof:

We know that,

$$|x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

$\Rightarrow |x| = max(x, -x) < k$

$\Leftrightarrow x < k \wedge -x < k$

$\Leftrightarrow x < k \wedge -k < x$

$\Leftrightarrow -k < x < k$

Examples

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Example

1. $< a_n >=< n^2 : n \in \mathbb{N} >=< n^2 >=< 1, 4, 9, ... >$,
   Range of the sequence $= \{1, 4, 9, ...\}$,
   This sequence is not bounded above but bounded below we can take 1,0,-1,-2,0.5 etc. as lower bounds. Since the sequence is bounded below but not bounded above so it is not a bounded sequence.

2. $(a_n) = (2 : n \in \mathbb{N}) = (2, 2, 2, ...)$,
   Range of the sequence $= \{2\}$.
   This sequence is bounded above as well as bounded below we can take 1,2,0,-1,-2,0.5 etc. as lower bounds and 2,3,4 etc as uper bounds. Since the sequence is bounded below as well as bounded above so it is a bounded sequence.

Examples

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Example

1. $< a_n >=< (-1)^n : n \in \mathbb{N} >=< -1, 1, -1, 1, ... >$,
   Range of the sequence $= \{-1, 1\}$.
   Explain about boundedness of the sequence?

2. $< a_n >=< \frac{1}{n} : n \in \mathbb{N} >=< 1, \frac{1}{2}, \frac{1}{3}, ... >$,
   Range of the sequence $= \{1, \frac{1}{2}, \frac{1}{3}, ...\}$.
   Explain about boundedness of the sequence?

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Definition

A sequence $< a_n >$ is said to converge to a real number l (or to have the real number l as its limit) if for every given $\epsilon > 0$, there exists a positive integer m (depending on ϵ) such that $|a_n - l| < \epsilon, \forall n \geq m$.
The same thing expressed in symbol is $a_n \to l$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = l$.

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Convergence of Sequences with Some Examples

Example

Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof:- Let $a_n = \frac{1}{n}$. Let ϵ be any positive number.
Therefore, $|a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$, if $n > \frac{1}{\epsilon}$.
Let m be a positive integer $m > \frac{1}{\epsilon}$. Thus $|a_n - 0| < \epsilon$, $\forall n \geq m$.
Hence $\lim_{n\to\infty} \frac{1}{n} = 0$.

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Example

Show that the constant sequence $< 2, 2, \ldots >$ converges to 2.
Proof:- Let $a_n = 2$. Let ϵ be any positive number.
Therefore, $|a_n - 2| = 0 < \epsilon$, $\forall n \geq 1$.
Let m be any positive integer. Thus $|a_n - 0| < \epsilon$, $\forall n \geq m$.
Hence $\lim_{n\to\infty} 2 = 2$.

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Example

Show that $\lim_{n\to\infty} \frac{1+2+\ldots+n}{n^2} = \frac{1}{2}$.

Proof:- Let $a_n = \frac{1+2+\ldots+n}{n^2} = \frac{n(n+1)}{2n^2}$. Let ϵ be any positive number.

Therefore, $|a_n - \frac{1}{2}| = |\frac{1}{2} + \frac{1}{2n} - \frac{1}{2}| = \frac{1}{2n} < \epsilon$, if $n > \frac{1}{2\epsilon}$.

Let m be a positive integer $m > \frac{1}{2\epsilon}$. Thus $|a_n - \frac{1}{2}| < \epsilon$, $\forall n \geq m$.

Hence proved.

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THANK YOU