Vector Analysis

Some important definitions about vectors:

① **Unit vector:** A vector of unit magnitude is called a unit vector. A unit vector in the direction of $\vec{A}$ is written as $\hat{A}$ read as A, hat or A. caret.

$\vec{A} = \hat{A} A$ or $\vec{A}a$ (a → direction is sometimes also represented by bold faced small letter a)

② **Zero or Null Vector:** It is a vector of zero magnitude denoted by 0. For null vector initial and terminal points are coincident.

③ **Proper Vectors:** Vectors other than null vectors are called proper vectors.

④ **Negative vector:** A vector having same magnitude but direction opposite to that of a given vector is called negative vector relative to that vector, as shown in figure 1.

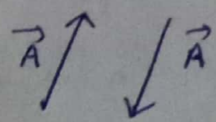


fig: 1

⑤ **Co-Initial Vectors:** Vectors are called co-initial when they have common initial point.

⑥ **Collinear / Parallel Vectors:** Those vectors, which have the same line of action or lines of action parallel to the same line are called collinear vectors.

Eg: $\vec{A}$ and $\vec{C}$ (same line of action)
$\vec{A}$ and $\vec{B}$ (parallel line of action)

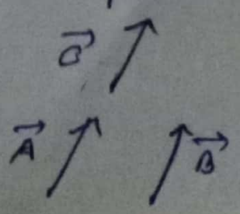


fig: 2

(7) Equality of Vectors:

Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal, if they have same magnitude and direction. For example, if $\vec{A}$ and $\vec{B}$ are equal vectors, then

$$\vec{A} = \vec{B}$$

Multiplication and Division of a vector by a Scalar:

The product $n\vec{A}$ or $\vec{A}n$ of a vector $\vec{A}$ and a scalar n is defined as a vector whose magnitude is equal to $n|A|$, and whose direction is the same as that of $\vec{A}$. The relation $\vec{B} = n\vec{A}$ is represented by the arrow whose length is n times the length of the vector $\vec{A}$ and direction is parallel to $\vec{A}$.

If 'n' is a scalar,

$$n(\vec{A}+\vec{B}) = n\vec{A} + n\vec{B}$$

$$(n+m)\vec{A} = n\vec{A} + m\vec{A}$$

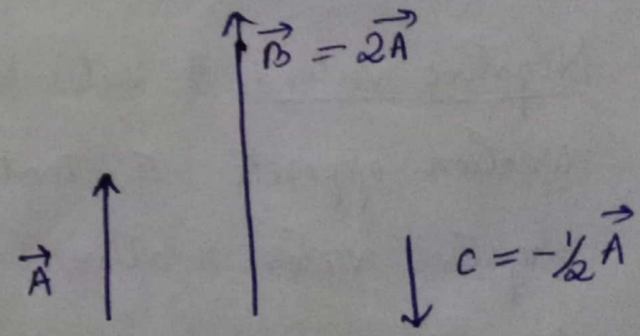


fig: 3

The division of a vector $\vec{A}$ by a non-zero scalar 'm' is simply the multiplication of a vector $\vec{A}$ by $\frac{1}{m}$

Addition and Subtraction of two vectors:

(1) Addition:

(a) The Parallelogram law of vector addition:

According to this law, the sum / resultant $\vec{R}$ of two vectors $\vec{A}$ and $\vec{B}$ is the diagonal of the parallelogram of which $\vec{A}$ and $\vec{B}$ are two adjacent sides. $\vec{R} = \vec{A} + \vec{B}$.

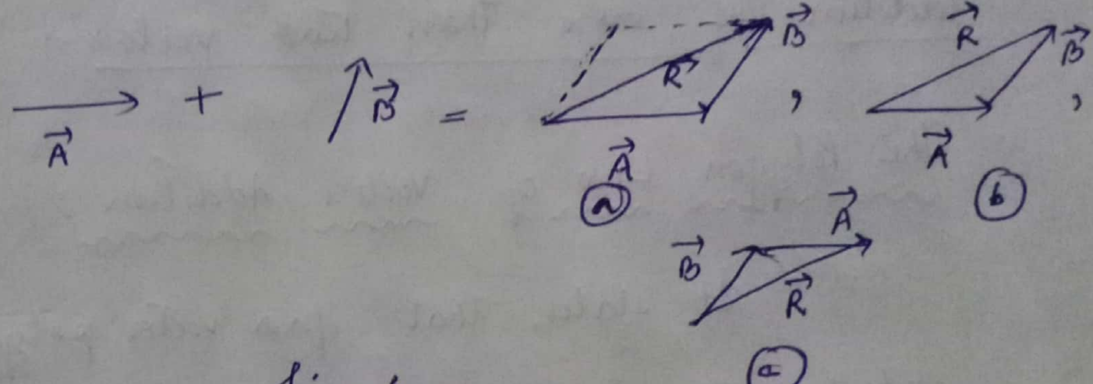


fig: 4

(b) The Triangle law of vector addition:

Let $\vec{A}$ and $\vec{B}$ be two vectors. For obtaining their resultant / sum, $\vec{R}$; $\vec{B}$ is carried parallel to itself until the tail of $\vec{B}$ coincides with the head of $\vec{A}$. $\vec{R} = \vec{A} + \vec{B}$ as shown in figure 4 (b).

(2) Subtraction :

Vectors are subtracted using negative vectors. The negative vector $-\vec{A}$ is defined as the vector whose magnitude is the same as that of $\vec{A}$ but direction opposite to the direction of $\vec{A}$.

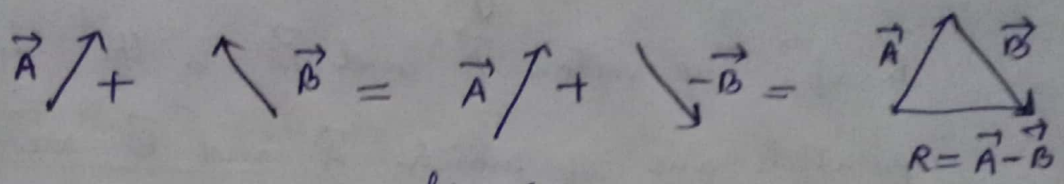
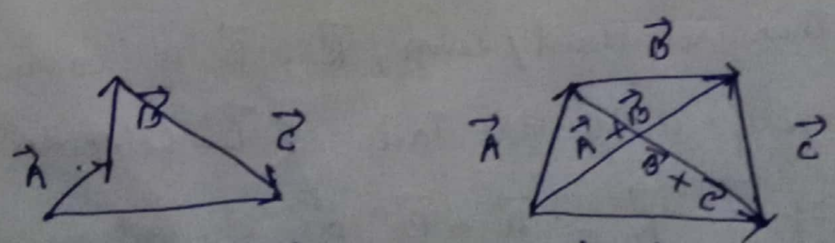


fig: 5

Addition of more than two vectors :

The Polygon Law of Vectors' Addition :

It states that if a vector polygon be drawn, placing the tail end of each succeeding vector at the head of the preceding one. Then their resultant $\vec{R}$ is drawn from the tail-end of the first to the head-end of the last as shown in figure 6.



$\vec{R} = \vec{A} + \vec{B} + \vec{C}$

$\vec{R} = \vec{A} + (\vec{B} + \vec{C})$
$= (\vec{A} + \vec{B}) + \vec{C}$.

fig: 6

(Vector addition is also associative)

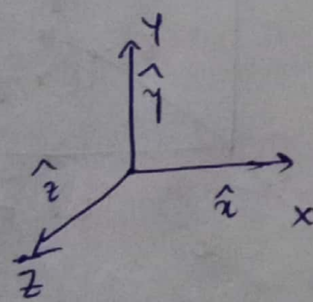
Components of a Vector :

A vector $\vec{A}$ can be decomposed into any number of vectors whose sum is $\vec{A}$. The vectors obtained after decomposition are called components of vectors $\vec{A}$.

It is often easier to set up Cartesian co-ordinates x, y, z and work with vector components. Let $\hat{i}, \hat{j}, \hat{k}$ (or $\hat{x}, \hat{y}, \hat{z}$) be unit vectors parallel to the x, y, z axes respectively. Then,

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}.$$

A_x, A_y, A_z are called components of A; geometrically, they are the projections of A along the three co-ordinate axes.



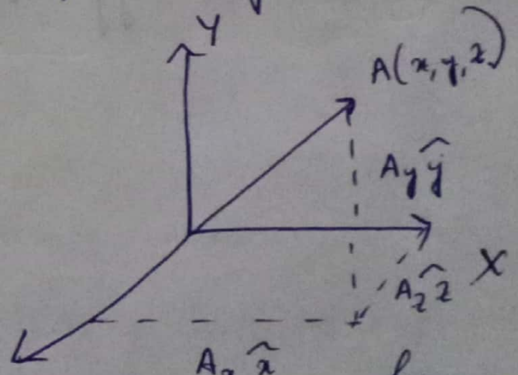


fig. 7

$$\vec{A} = \vec{A_x} + \vec{A_y} + \vec{A_z}$$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}.$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$ (By Pythagoras theorem in three dimension)

The unit vector $\hat{A}$ in the direction of $\vec{A}$ is given by,

$$\hat{A} = \frac{\vec{A}}{A} = \frac{A_x}{A}\hat{x} + \frac{A_y}{A}\hat{y} + \frac{A_z}{A}\hat{z}$$

$\frac{A_x}{A}, \frac{A_y}{A}, \frac{A_z}{A}$ are the three direction cosines of $\vec{A}$

Position and Displacement Vector:

The location of a point in 3-D can be described by listing its cartesian co-ordinates (x, y, z). The vector to that point from the origin is called the position vector:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

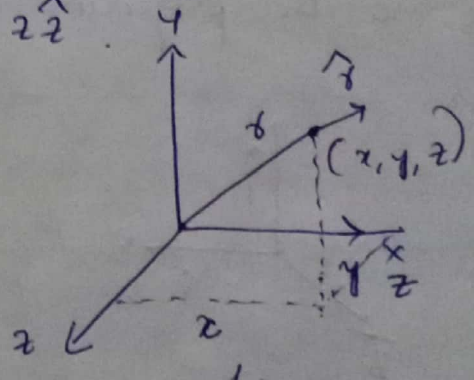
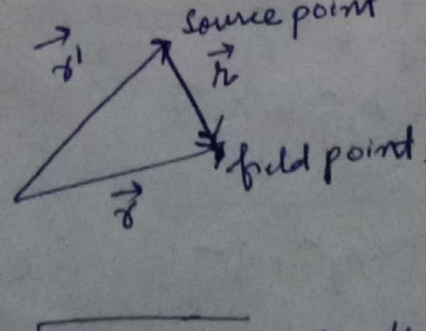


fig 8



The magnitude, $r = \sqrt{x^2+y^2+z^2}$ is the distance from origin.

$$\& \ \hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$ is a unit vector pointing radially outward.

The displacement vector $\vec{dr}$ from (x, y, z) to $(x+dx, y+dy, z+dz)$ is

$$\vec{dr} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}.$$

In Electromagnetics, we generally encounter such problems involving two points — typically a source point r', where an electric charge is located, and a field point r where we are calculating the electric or magnetic field.

$$\mathscr{r} = r - r'$$

$$|\mathscr{r}| = |r - r'| \;, \quad \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\mathscr{r}|} = \frac{r - r'}{|r - r'|}$$

In Cartesian co-ordinates,

$$\mathscr{r} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}.$$

$$|\mathscr{r}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

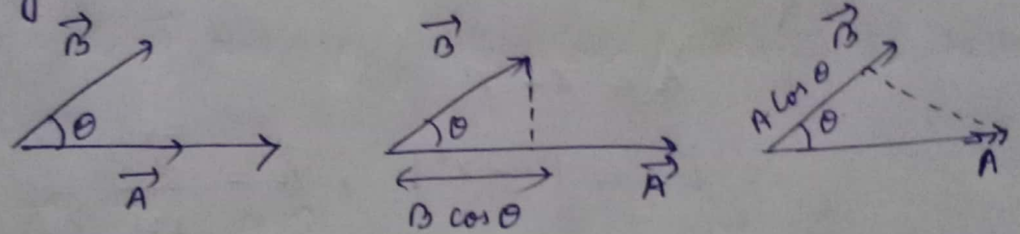
$$\hat{\mathscr{r}} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

Product of Two Vectors

① Scalar / Dot product of two Vectors:

Denoted by $\vec{A}\cdot\vec{B}$ for two vectors $\vec{A}$ and $\vec{B}$ and read as $\vec{A}$ dot $\vec{B}$.

It is defined as a scalar equal to the product of the magnitudes of these vectors multiplied by the cosine of the angle between them as shown in figure 9.



$$\vec{A}\cdot\vec{B} = AB\cos\theta = A(B\cos\theta) = B(A\cos\theta)$$

(Dot product of $\vec{A}$ and $\vec{B}$ is either the multiplication of the magnitude of $\vec{A}$ and the projection of $\vec{B}$ in the direction of $\vec{A}$ or vice-versa)

Dot product possesses distributive property, i.e,

$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ for three vectors $\vec{A},\vec{B},\vec{C}$.

$\vec{A}\cdot\vec{A} = A^2$.

$$\left\{\begin{array}{ll} \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 & \text{(angle is zero)} \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 & \text{(angle is } \pi/2\text{)} \end{array}\right\}$$

$\downarrow$

dot product of cartesian unit vectors.

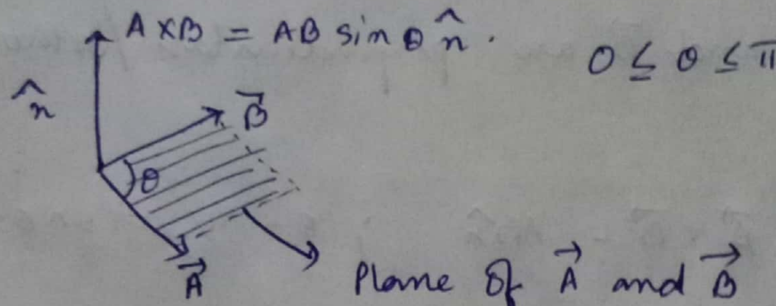
$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z.$$

② Vector / Cross Product

Denoted by $\vec{A} \times \vec{B}$, read as $\vec{A}$ cross $\vec{B}$. It is defined as a vector $\vec{R}$ whose magnitude is equal to the product of the magnitude of two vectors multiplied by the sine of the angle between them, and directed along a unit vector $\hat{n}$, which is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Thus,
$$\vec{R} = \vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$ as shown in figure

$$0 \le \theta \le \pi$$



*** Imp. Points:**

ⓐ Since $0 \le \theta \le \pi$, $\sin\theta$ cannot be negative, hence $|\vec{R}|$ or R cannot be negative.

ⓑ Vector product is not commutative.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A} = -\vec{B} \times \vec{A} \quad \left[\vec{B} \times \vec{A} = -AB\sin\theta \right.$$

ⓒ Vector product is distributive, thus

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

ⓓ Vector product is associative.

$$\vec{A} \times (m\vec{B}) = (m\vec{A}) \times \vec{B} = m\, AB \sin\theta \, \hat{n}.$$

(e) For collinear vectors: $\theta = 0 / \pi$. Thus $\sin\theta = 0$,

Then
$$\vec{A} \times \vec{B} = AB \sin\theta \,\hat{n} = 0.$$

For equal vectors,
$$\vec{A} \times \vec{A} = 0.$$

For unit vectors:
$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0.$$

(f) If $\vec{A}$ and $\vec{B}$ are perpendicular/orthogonal to each other,
$$\vec{A} \times \vec{B} = AB\hat{n} \quad ; \theta = \pi/2 \Rightarrow \sin\theta = 1.$$

Thus, for mutually perpendicular unit vectors
$$\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$$
$$\hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j}$$
$$\hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}.$$

(g) In terms of components,
$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}.$$
$$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
$$\because \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0.$$
$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} \text{ etc.}$$

Thus,

$$\vec{A}\times\vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - B_z A_x)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

It can be easily remembered using the determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Applications :

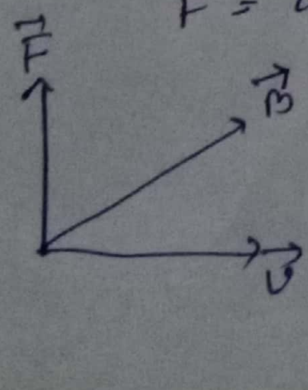
① Force on a moving charge in a magnetic field.

If a ~~f~~ charge 'q' is moving with a velocity $\vec{v}$ at an angle θ with a magnetic field $\vec{B}$. Then the force acting on the charged particle is given by the relation

$$\boxed{\vec{F} = qvB\sin\theta}$$

where $\vec{F}$, v and B are the magnitudes of the force, velocity and magnetic field respectively. In vector form.

$$\vec{F} = q(\vec{v}\times\vec{B}) \qquad \text{(Lorentz force)}$$



(1) Vector area:

$\vec{A} \times \vec{B} = AB \sin\theta$, i.e. area of the parallelogram with sides $\vec{A}$ and $\vec{B}$ inclined at angle θ.

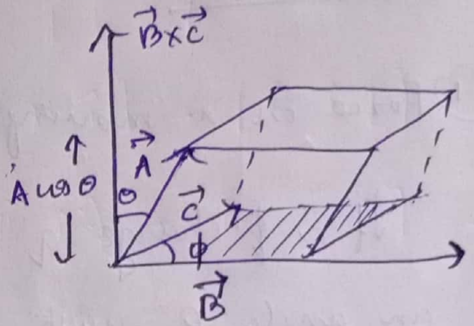
Triple Product:

(1) Scalar Triple Product, $\vec{A} \cdot (\vec{B} \times \vec{C})$

(2) Triple Cross or Triple vector product $\vec{A} \times (\vec{B} \times \vec{C})$

(1) Scalar Triple Product:

fig 12:



Let the vectors $\vec{A}, \vec{B}, \vec{C}$ represents the edges of a parallelopiped as shown in figure 12. $(\vec{B} \times \vec{C})$ is a vector normal to the plane of $\vec{B}$ and $\vec{C}$.

The area of the base of the parallelopiped $= BC \sin\phi$. i.e $\vec{B} \times \vec{C}$ and its direction is perpendicular to $\vec{B}$ & $\vec{C}$ as shown in figure.

The scalar product of $\vec{A}$ with $(\vec{B} \times \vec{C})$ is the product of this area and the projection of $\vec{A}$ along $(\vec{B} \times \vec{C})$ i.e $A \cos\theta$.

$\therefore \vec{A} \cdot (\vec{B} \times \vec{C}) = A \cos\theta \, (\vec{B} \times \vec{C})$

$\vec{A}\cdot(\vec{B}\times\vec{C})$ = Vertical height of parallelopiped x area of its base

= Volume of the parallelopiped.

Any face can be taken as the base and hence its volume can be represented by the following three expressions,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

To retain the volume with positive sign, the cyclic order of $\vec{A}$, $\vec{B}$ and $\vec{C}$ should be maintained because, we know, $\vec{B}\times\vec{C} = -\vec{C}\times\vec{B}$. As the order of terms in scalar product is meaningless, we have.

$$(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A} = (\vec{C}\times\vec{A})\cdot\vec{B}$$

The value of the scalar triple product depends on the cyclic order of the vectors and is independent of the position of the dots and crosses

Scalar triple product is often written as $[\vec{A}\ \vec{B}\ \vec{C}]$

Important points about scalar triple product:

1. $[\vec{A}\ \vec{B}\ \vec{C}] = \vec{A}\cdot(\vec{B}\times\vec{C})$

or $[\vec{A}\ \vec{B}\ \vec{C}] = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z\end{vmatrix}$

$$= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)$$

$$[\vec{A}\ \vec{B}\ \vec{C}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(2) For unit vectors $\hat{i}, \hat{j}, \hat{k}$, we have

$$[\hat{i}\ \hat{j}\ \hat{k}] = (\hat{i} \times \hat{j}) \cdot \hat{k}$$

$$\Rightarrow [\hat{i}\ \hat{j}\ \hat{k}] = \hat{k} \cdot \hat{k}$$

$$\Rightarrow [\hat{i}\ \hat{j}\ \hat{k}] = 1$$

Similarly, $[\hat{j}\ \hat{k}\ \hat{i}] = [\hat{k}\ \hat{i}\ \hat{j}] = 1$

Hence, $[\hat{i}\ \hat{j}\ \hat{k}] = [\hat{j}\ \hat{k}\ \hat{i}] = [\hat{k}\ \hat{i}\ \hat{j}] = 1$

But, $[\hat{i}\ \hat{k}\ \hat{j}] = [\hat{k}\ \hat{j}\ \hat{i}] = [\hat{j}\ \hat{i}\ \hat{k}] = -1.$

(3) For 2 equal vectors among the 3, the scalar triple product is 0.

eg: $[\vec{A}\ \vec{A}\ \vec{B}] = (\vec{A} \times \vec{A}) \cdot \vec{B} = 0$ $[\because \vec{A} \times \vec{A} = 0]$

(4) If 2 vectors are parallel, then their scalar triple product is 0. If $\vec{A}$ and $\vec{B}$ be parallel, we have

$\vec{B} = n\vec{A}$ where n is some scalar.

Thus, we have,

$$[\vec{A}\ \vec{B}\ \vec{C}] = [\vec{A} \times \vec{B}] \cdot \vec{C} = (\vec{A} \times n\vec{A}) \cdot \vec{C}$$

$$= n(\vec{A} \times \vec{A}) \cdot \vec{C}$$

② Vector Triple Product $\vec{A} \times (\vec{B} \times \vec{C})$

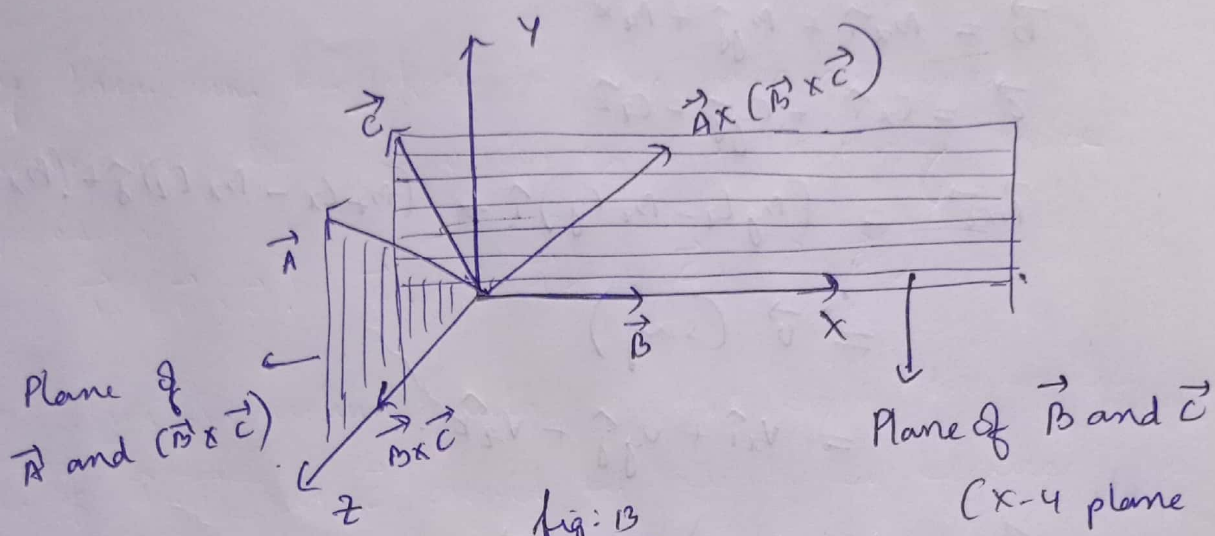


fig: 13

Cross product $(\vec{B} \times \vec{C})$ is a vector normal to the plane containing $\vec{B}$ and $\vec{C}$. Therefore, vector $\vec{A} \times (\vec{B} \times \vec{C})$ will be perpendicular to the plane containing $\vec{A}$ and $(\vec{B} \times \vec{C})$ i.e it will lie in the plane of $\vec{B}$ and $\vec{C}$ as shown in figure 13.

Similarly, vector $\vec{B} \times (\vec{A} \times \vec{C})$ lies in the plane of $\vec{A}$ and $\vec{C}$ and is perpendicular to $\vec{B}$ and $\vec{C} \times (\vec{A} \times \vec{B})$ lies in the plane of $\vec{A}$ and $\vec{B}$ and is perpendicular to $\vec{C}$. Thus we see that

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}.$$

Thus, vector multiplication is not associative.

$A \times (\vec{B} \times \vec{C})$

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$.

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$.

$\vec{B}\times\vec{C} = (B_y C_z - B_z C_y)\hat{i} + (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k}$

$= \vec{v}$ (say)

$= v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$.

Then, $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{A}\times\vec{v}$

$= (A_y v_z - v_y A_z)\hat{i} + (A_z v_x - A_x v_z)\hat{j} +$

$(A_x v_y - A_y v_x)\hat{k}$.

Its x-component $= A_y(B_x C_y - B_y C_x) - A_z \; (B_z C_x - B_x C_z)]\hat{i}$

$= [A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z C_z B_x]\hat{i}$

On adding and subtracting $A_x C_x B_x \hat{i}$ on RHS, we get.

A_y ~~$C_y B_x$ +~~

$[A_y C_y B_x + A_z C_z B_x + A_x C_x B_x - A_y B_y C_x -$

$A_z B_z C_x - A_x B_x C_x]\hat{i}$

$= (\vec{A}.\vec{C}) B_x\hat{i} - (\vec{A}.\vec{B}) C_x\hat{i}$.

Similarly, y-component $= (\vec{A}\cdot\vec{C})\, B_y \hat{j} - (\vec{A}\cdot\vec{B})\, C_y \hat{j}$

z-component $= (\vec{A}\cdot\vec{C})\, B_z \hat{k} - (\vec{A}\cdot\vec{B})\, C_z \hat{k}$,

Hence,

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\,(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) - (\vec{A}\cdot\vec{B})\,(C_x\hat{i} + C_y\hat{j} + C_z\hat{k})$$

$$= (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{A}\cdot\vec{B})\,C.$$

Changing the order of multiplication in a vector triple product changes its value as shown here,

$$\left(\vec{A}\times\{\vec{B}\times\vec{C}\}\right) = -\vec{C}\times(\vec{A}\times\vec{B}) = (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{B}\cdot\vec{C})\,\vec{A}.$$